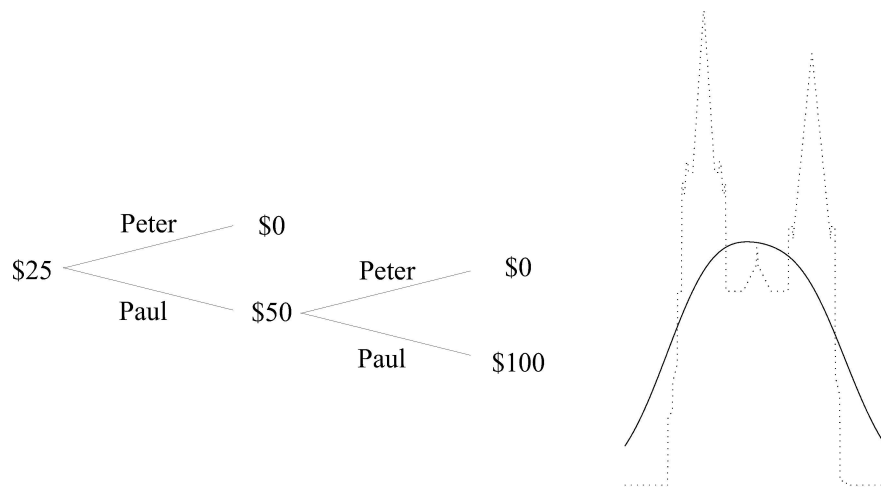


The universal measure of nonstochastic objects

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Abstract

The usual definitions of stochasticity are very different from Kolmogorov's original definition as recorded by Shen in November 1981. This note simplifies Kolmogorov's definition and shows that the universal machine generates non-stochastic objects with surprisingly high probability.

A version of this note is published as an arXiv technical report: <http://arxiv.org/abs/2608.27098>.

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1 Introduction

The notion of stochastic objects was first introduced (in the currently recorded history) in Kolmogorov’s talk at a seminar at Moscow State University on 26 November 1981. In the same talk he also posed the problem of estimating the abundance of stochastic and nonstochastic objects. (Alexander Shen’s notes from that seminar have been published as [6, note 5.12].) Various solutions to this problem have been given by Shen in 1983 [7], Vladimir V’yugin in 1985 [11], Andrei Muchnik in 1998 [5, Theorems 10.10 and 10.11], and Shen et al. more recently [8, Theorem 251]. See [8, Sect. 14.2] and [9, Sect. 2] for excellent reviews.

A surprising feature of the existing literature on stochastic objects is that it always assumes some externally given nested structure on the objects whose stochasticity we are interested in. In many cases the objects are the binary strings, with their lengths playing an important role. In other cases (e.g., in [7]), the objects are the nonnegative integers with the nested structure consisting of the sets $\{0, \dots, 2^n - 1\}$, $n = 0, 1, \dots$. Kolmogorov’s original problem [6, Sect. 5.12] is completely different: no externally given nested structure is used in measuring sets of nonstochastic objects. Kolmogorov’s setting will be discussed in the appendix. This note simplifies it further. While Kolmogorov’s setting still involves a nested structure, albeit intrinsic, this note dispenses with it altogether.

The standard problem of measuring the set of all (α, β) -nonstochastic objects (where α and β are parameters, to be defined in Sect. 2) has an important special case: do such objects even exist in a given level of the nested structure? (Such as the question of existence of (α, β) -nonstochastic strings of length n .) Important results in this direction are due to Shen [7, Theorem 2] and Gács et al. [4, Theorem IV.2]. Such problems do not arise in the setting of this note.

Different authors and publications use different ways of measuring sets of nonstochastic objects. Shen [7, Theorem 3] bounds the uniform probability measure of nonstochastic objects. Considering the uniform probability measure appears somewhat unnatural, and [7, Theorem 4] covers non-uniform probability measures; Shen’s Theorem 4, however, only gives a one-sided bound. V’yugin [11, Corollary and Theorem 3] bounds the universal semimeasure of nonstochastic objects. The existing results that are closest to the main result of this note are Theorem 10.11 in [5] and Theorem 251 in [8, Sect. 14.2]; both these results involve a different version of the universal semimeasure. In the last three of these results, those in [11, 5, 8], the bounds on the measure of the (α, β) -nonstochastic objects are governed by the term $2^{-\alpha}$. Kolmogorov’s setting described in the appendix is very different, but we can still expect a similar prominent role for $2^{-\alpha}$. The nature of the main result of this note is even more different, and $2^{-\alpha}$ disappears.

Let $\mathbb{N} := \{1, 2, \dots\}$ be the natural numbers, $[m] := \{1, \dots, m\}$ for each $m \in \mathbb{N}$, $\log$ be binary logarithm, C be plain Kolmogorov complexity [8, Sect. 1.1], K be prefix complexity [8, Sect. 4.3], and m be a fixed universal semimeasure on $\mathbb{N}$ [8, Sect. 4.2]. For a subset $A \subseteq \mathbb{N}$, we set $m(A) := \sum_{a \in A} m(a)$ (therefore,

$m(\{a\}) = m(a)$ for all $a \in \mathbb{N}$). This makes m a measure, so let us also refer to it as the *universal measure*.

For a brief summary of the main results about finitary randomness, including connections with universal p-values and e-values, see [10, Appendix A].

2 Main result

Let us refer to elements of $\mathbb{N}$ as objects (we can replace $\mathbb{N}$ by any set in computable bijection with $\mathbb{N}$, such as the binary strings $\{0, 1\}^*$). An object $\omega \in \mathbb{N}$ is (α, β) -*stochastic*, where α and β are nonnegative integers, if there exists a finite set $\Omega \subseteq \mathbb{N}$ for which

- $\omega \in \Omega$;
- $K(\Omega) \leq \alpha$;
- $K(\omega \mid \Omega) \geq \log |\Omega| - \beta$.

Otherwise ω is (α, β) -*nonstochastic*. In our notation we follow Kolmogorov [6, Sect. 5.12]; for differences from Kolmogorov's setting, see the appendix.

Let $\mathcal{N}_{\alpha, \beta} \subseteq \mathbb{N}$ be the set of all (α, β) -nonstochastic objects. We are interested in the asymptotics of $m(\mathcal{N}_{\alpha, \beta})$ as $\alpha, \beta \rightarrow \infty$. Let

$$\tau(t) := m(\{t+1, t+2, \dots\})$$

be the tail of m .

Theorem 1. *There exists $c > 1$ such that, for all α and $\beta \leq 2^{2^\alpha}$,*

$$\frac{\tau(\alpha)}{c} \leq m(\mathcal{N}_{\alpha, \beta}) \leq c\tau(\alpha). \quad (1)$$

More generally, let us define $\text{tower}_k : [0, \infty) \rightarrow [0, \infty)$ recursively by

$$\text{tower}_i(\alpha) := \begin{cases} \alpha & \text{if } i = 0 \\ 2^{\text{tower}_{i-1}(\alpha)} & \text{for } i = 1, 2, \dots \end{cases}$$

The condition $\beta \leq \text{tower}_2(\alpha)$ in the statement of the theorem can be replaced by $\beta \leq \text{tower}_k(\alpha)$ for a fixed k ("fixed" in the sense of c depending on k).

The theorem says that $m(\mathcal{N}_{\alpha, \beta}) = \Theta(\tau(\alpha))$ under mild conditions on β . Informally, nonstochastic objects are generated with probability that decays extremely slowly. The function $n \mapsto \tau(n)$ monotonically decreases to 0 but does so more slowly than any computable function monotonically decreasing to 0, such as $1/\log \log n$ (and we can take any number of logarithms). Let us state this observation formally.

Lemma 2. *Let $f : \mathbb{N} \rightarrow (0, \infty)$ be a computable monotonically decreasing function such that $f(n) \rightarrow 0$ as $n \rightarrow \infty$. Then $\tau(n)/f(n) \rightarrow \infty$ as $n \rightarrow \infty$.*

Proof. It suffices to prove that any computable positive series $\sum_{n=1}^{\infty} a_n \leq 1$ with computable tails can be transformed into a computable nonnegative series $\sum_{n=1}^{\infty} a'_n \leq 1$ such that $a'_n/a_n \rightarrow \infty$. (This is true even if “monotonically decreasing” is understood in the wide sense allowing intervals of constancy.) To obtain a'_n , we will divide the natural numbers $\mathbb{N}$ into disjoint finite non-empty intervals denoted by $I_0, I_1, \dots$ in the ascending order and modify a_n for $n \in I_k$ sequentially for $k = 0, 1, \dots$. Start with defining I_0 arbitrarily and setting

$$a'_n := \begin{cases} 0 & \text{if } n \in I_0 \\ a_n & \text{otherwise.} \end{cases}$$

Divide the “overall budget” $c := \sum_{n \in I_0} a_n$ into a positive series $c = \sum_{k \in \mathbb{N}} c_k$. Take I_1 so large that $\sum_{n \in I_2 \cup I_3 \cup \dots} a'_n < c_1$ and redefine $a'_n := 2a'_n$ for all $n \in I_2 \cup I_3 \cup \dots$. Take I_2 so large that $\sum_{n \in I_3 \cup I_4 \cup \dots} a'_n < c_2$ and redefine $a'_n := 2a'_n$ for all $n \in I_3 \cup I_4 \cup \dots$, etc. It is clear that for each n the value of a'_n will eventually stabilize and that the resulting series will satisfy $a'_n/a_n \rightarrow \infty$ as $n \rightarrow \infty$. $\square$

Theorem 1 continues to hold when prefix complexity K is replaced by plain Kolmogorov complexity C . Let us call the statement about C the C -version of Theorem 1; let us also apply the expression “ C -version” in the analogous sense to other statements in the rest of the note.

3 Proof of Theorem 1

In this proof, $c_1, c_2, \dots$ stand for absolute positive constants. The C -version of the theorem will be treated in parallel.

Let us start with the easy upper bound in (1). Every $\omega \in \mathcal{N}_{\alpha, \beta}$ satisfies $K(\omega) \geq \alpha - c_1$ (otherwise $\Omega := \{\omega\}$ would have witnessed its $(\alpha, 0)$ -stochasticity). Therefore, every $\omega \in \mathcal{N}_{\alpha, 0}$ satisfies $2 \log \omega \geq \alpha - c_2$, which implies

$$\omega \geq 2^{\alpha/2 - c_2/2} \geq \alpha + 1 \quad (2)$$

for all $\alpha \geq c_3$. This implies

$$m(\mathcal{N}_{\alpha, \beta}) \leq m(\mathcal{N}_{\alpha, 0}) \leq m(\{\alpha + 1, \alpha + 2, \dots\}) = \tau(\alpha)$$

from $\alpha = c_3$ on. (This is sufficient since the remaining finitely many α s can be absorbed into c .)

It remains to prove the lower bound. Let us first establish the great stability of τ ; it hardly changes over huge regions of its domain. (This is why we will be able to get away with the extremely crude second inequality in (2).) For that, we will need the following lemma, which, roughly, bounds m from below by uniform distributions.

Lemma 3. *For every $A \subseteq \mathbb{N}$,*

$$m(A) \geq c_4 \sum_{n \in \mathbb{N}} m(n) 2^{-n} |A \cap [2^n]|.$$

Proof. Being the universal measure, m dominates the mixture of the uniform probability measures on $[2^n]$ taken with the weights $m(n)$. $\square$

Now we can establish the stability of τ .

Lemma 4. *For every $t \in \mathbb{N}$, $\tau(t) \geq c_5 \tau(\lceil \log t \rceil)$; in particular, $c_5 \tau(t) \leq \tau(2^t) \leq \tau(t)$.*

Proof. Let us apply Lemma 3 to $A := \{t+1, t+2, \dots\}$ with $|A \cap [2^n]| = (2^n - t)^+$. For $n \geq \lceil \log t \rceil + 1$, we have $2^n \geq 2t$, and therefore, $2^n - t \geq 2^{n-1}$; this gives

$$\tau(t) \geq c_4 \sum_{n=\lceil \log t \rceil+1}^{\infty} m(n) 2^{-n} 2^{n-1} = \frac{c_4}{2} \tau(\lceil \log t \rceil). \quad \square$$

The next key observation (cf. [4, Lemmas III.8 and III.10]) is that the complexity of stochastic objects is simple.

Lemma 5. *Every (α, β) -stochastic $\omega \in \mathbb{N}$ satisfies*

$$K(K(\omega)) \leq \alpha + 2 \log(\alpha + \beta + 2) + c_6.$$

Proof. The proof uses the fact that $K(\omega) \approx \log |\Omega|$:

$$\begin{aligned} K(\omega) &\leq K(\omega \mid \Omega) + K(\Omega) + c_7 \leq \log |\Omega| + \alpha + c_8 \\ K(\omega) &\geq K(\omega \mid \Omega) - c_9 \geq \log |\Omega| - \beta - c_9, \end{aligned} \quad (3)$$

where Ω witnesses the (α, β) -stochasticity of ω . Since $\log |\Omega|$ is computable given Ω ,

$$K(K(\omega)) \leq K(\Omega) + 2 \log(\alpha + \beta + 2) + c_{10} \leq \alpha + 2 \log(\alpha + \beta + 2) + c_6. \quad \square$$

Lemma 5 shows that nonstochastic objects ω are plentiful: it is enough for ω to live at a complex complexity level.

Corollary 6. *Set $\alpha' := \lceil \alpha + c_{11} \log(\alpha + \beta + 2) \rceil$ and let $n \in \mathbb{N}$ be such that $K(n) > \alpha'$. Then at least 2^{n-1} of the numbers in $[2^n]$ are (α, β) -nonstochastic.*

Proof of the C-version of Corollary 6. The proof for the C -version is simpler and more intuitive, so let us start with it. At the end of the section we will see how it can be modified to cover the original version (the K -version). There we will also check the C -version of Lemma 5.

Let $\omega \in [2^n]$ satisfy $C(\omega) \geq n - 1$ (there are at least 2^{n-1} such ω). The difference $n - C(\omega)$ takes one of finitely many values, so $C(C(\omega)) \geq C(n) - c_{12} > \alpha' - c_{12}$; therefore, by the C -version of Lemma 5, ω is (α, β) -nonstochastic (for a suitable choice of the absolute constants). $\square$

The proof shows that, for such an n , a random number in $[2^n]$ is nonstochastic. Its obvious model $\Omega := [2^n]$ costs about $C(n)$ bits, which exceeds the budget α , and any cheaper model would, by Lemma 5, describe n too cheaply.

Now we can make Corollary 6 into a lower bound on $m(\mathcal{N}_{\alpha, \beta})$.

Corollary 7. For all α and β ,

$$m(\mathcal{N}_{\alpha,\beta}) \geq c_{13}\tau(\alpha'), \quad (4)$$

where α' is as defined in Corollary 6.

Proof. By Corollary 6 (whose K -version is proved at the end of the section), every n with $K(n) > \alpha'$ satisfies $|\mathcal{N}_{\alpha,\beta} \cap [2^n]| \geq 2^{n-1}$; therefore, Lemma 3 applied to $A := \mathcal{N}_{\alpha,\beta}$ gives

$$m(\mathcal{N}_{\alpha,\beta}) \geq c_4 \sum_{n:K(n)>\alpha'} m(n)2^{-n}2^{n-1} = \frac{c_4}{2} m(\{n : K(n) > \alpha'\}).$$

Now let us apply Lemma 3 to $A := \{n : K(n) > \alpha'\}$. At most $2^{\alpha'}$ numbers n have $K(n) \leq \alpha'$, so for $k \geq \alpha' + 1$ we have $|A \cap [2^k]| \geq 2^k - 2^{\alpha'} \geq 2^{k-1}$; therefore,

$$m(\{n : K(n) > \alpha'\}) \geq c_4 \sum_{k=\alpha'+1}^{\infty} m(k)2^{-k}2^{k-1} = \frac{c_4}{2} \tau(\alpha'). \quad \square$$

Now we can prove the lower bound in (1). First note that $\beta \leq 2^{2^\alpha}$ implies

$$\alpha' = \lceil \alpha + c_{11} \log(\alpha + \beta + 2) \rceil \leq 2^{2^\alpha}$$

for large α . Therefore, by Corollary 7, the monotonicity of τ , and two applications of Lemma 4,

$$m(\mathcal{N}_{\alpha,\beta}) \geq c_{13}\tau(\alpha') \geq c_{13}\tau(2^{2^\alpha}) \geq c_{13}c_5\tau(2^\alpha) \geq c_{13}c_5^2\tau(\alpha).$$

This can be extended to any fixed tower $_k$ by applying Lemma 4 more times.

Now let us establish the C -version of Theorem 1. As the first step, replace all occurrences of K by C . In the proof of the C -version of Lemma 5 we should also replace (3) by

$$C(\omega) \leq C(\omega \mid \Omega) + 2C(\Omega) + c_{14} \leq \log |\Omega| + 2\alpha + c_{15},$$

while the rest of the proof still works.

In the C -version of Corollary 7, we can only claim that at most $2^{\alpha'+1}$ numbers n have $C(n) \leq \alpha'$. We have to start the summation in the last displayed equation in the proof from $k = \alpha' + 2$, and so (4) continues to hold with α' replaced by $\alpha' + 1$. The derivation of the lower bound in Theorem 1 still works.

It remains to check the K -version of Corollary 6. At most 2^{n-1} objects ω have $K(\omega) \leq n - 1$, so at least $2^n - 2^{n-1} = 2^{n-1}$ of the objects $\omega \in [2^n]$ have $K(\omega) \geq n$; fix such an ω . Describing n and then reading n further bits (the binary representation, perhaps with leading 0s, of an element of $[2^n]$ decreased by 1) is a description mode demonstrating $K(\omega) \leq K(n) + n + c_{16}$, so the

difference $d := K(\omega) - n$ satisfies $0 \leq d \leq K(n) + c_{16}$. As $n = K(\omega) - d$ is computable from $K(\omega)$ and d ,

$$K(n) \leq K(K(\omega)) + K(d) + c_{17} \leq K(K(\omega)) + 2\log(K(n) + 1) + c_{18}.$$

Let us make c_{11} large compared with c_6 in Lemma 5. Since $v \mapsto v - 2\log(v + 2)$ is increasing for large v , and since $\log(K(n) + 1) \leq \log(\alpha + \beta + 2) + c_{19}$ may be assumed (otherwise $K(n)$ is so large that the inequality below is immediate), we get

$$K(K(\omega)) \geq K(n) - 2\log(K(n) + 1) - c_{18} > \alpha + 2\log(\alpha + \beta + 2) + c_6;$$

therefore, ω is nonstochastic by Lemma 5.

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A Kolmogorov's problems

In the next paragraph I reproduce Kolmogorov's problems given in his talk on 26 November 1981 and briefly recorded by Shen [6, Sect. 5.12].

A finite object ω is (α, β) -stochastic if there exists a finite set Ω such that

- $\omega \in \Omega$;
- $K(\Omega) \leq \alpha$;
- $K(\omega) \geq \log |\Omega| - \beta$.

Let $N(k, \alpha, \beta)$ be the number of (α, β) -stochastic objects of complexity at most k . Typical cases are $\alpha, \beta = \epsilon k$ and $\alpha, \beta = \sqrt{k}$. What can we say about the asymptotics of $N(k, \alpha, \beta)/2^k$ as $k \rightarrow \infty$? Is it true that $N(k, \alpha, \beta)/2^k \rightarrow 0$, if k increases and $\alpha/k, \beta/k \rightarrow 0$?

It is very likely that by K Kolmogorov meant plain Kolmogorov complexity, nowadays usually denoted by C , while his questions also make sense for prefix complexity. It is also likely that there is a typo in the definition of $N(k, \alpha, \beta)$ and in fact it means the number of (α, β) -nonstochastic objects of complexity at most k .

Kolmogorov used the condition $K(\omega) \geq \log |\Omega| - \beta$ rather than $K(\omega | \Omega) \geq \log |\Omega| - \beta$, which we used in the main note. The suggestion to measure the randomness deficiency of ω as an element of Ω by $\log |\Omega| - K(\omega | \Omega)$ rather than $\log |\Omega| - K(\omega)$ is due to Per Martin-Löf (personal communication), and it leads to an elegant theory: see, e.g., Propositions 11 and 13 in the summary [10, Appendix 1].

Kolmogorov's last question, "Is it true that $N(k, \alpha, \beta)/2^k \rightarrow 0 \dots$?", should probably be replaced by "When is it true that $N(k, \alpha, \beta)/2^k \rightarrow 0$?" The examples of pairs (α, β) that he lists may be complemented by the more challenging case $\alpha, \beta = \log k$.

B The tail of the universal measure

First let us state separately and more completely the closeness between μ and σ defined by

$$\sigma(t) := m(\{n : K(n) > t\}).$$

Lemma 8. *For some absolute constant $c > 1$,*

$$\tau/c \leq \sigma \leq c\tau. \quad (5)$$

Proof. The lower bound in (5) was established at the end of the proof of Corollary 7. The upper bound follows from

$$\begin{aligned} \sigma(t) &= m(\{n : K(n) > t\}) \\ &\leq m(\{n : n > 2^{t-2\log t - c_{20}}\}) \leq m(\{n : n > t\}) = \tau(t), \end{aligned}$$

where the last inequality $\leq$ holds for all $t \geq c_{21}$. $\square$

We can rewrite (5) as $\sigma = \Theta(\tau)$. Another interesting connection is with the version of the busy beaver function as defined by Chaitin [2, Definition 5.3] (see also [8, Sect. 1.2.2]). However, here we only have a one-sided relation. Let us set

$$\mu(t) := \min\{K(n) : n > t\},$$

so that μ is, essentially, the inverse to the busy beaver function.

Lemma 9. *For some absolute constant $c > 0$,*

$$\tau \geq c2^{-\mu}.$$

Proof. The inequality follows from

$$\tau(t) = m(\{n : n > t\}) \geq m(n^*) \geq c_{22}2^{-K(n^*)} = c_{22}2^{-\mu(t)},$$

where $n^* > t$ is any number that achieves $K(n^*) = \mu(t)$. $\square$

Finally, τ gives the rate of convergence of computable approximations to Chaitin's number Ω ; it is a well-studied object in the theory of Ω . See [1, Sects. 2.2.2 and 3] for a review. For example, Lemma 2 is a trivial aspect of an elegant characterization of Omega numbers obtained by Downey, Hirschfeldt, and Nies [3] (see also [1, (7)]).