# Another example of duality between game-theoretic and measure-theoretic probability

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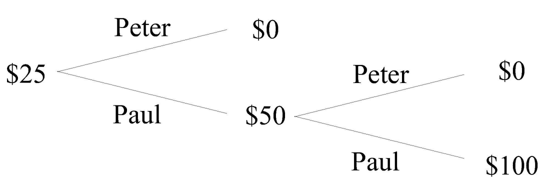


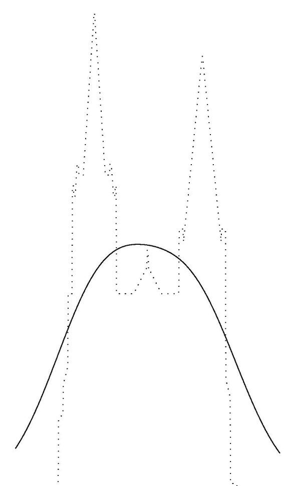

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## Abstract

This paper makes a small step towards a non-stochastic version of superhedging duality relations in the case of one traded security with a continuous price path. Namely, we prove the coincidence of game-theoretic and measure-theoretic expectation for lower semicontinuous positive functionals. We consider a new broad definition of game-theoretic probability, leaving the older narrower definitions for future work.

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# 1 Introduction

The words like "positive" and "increasing" will be understood in the wide sense (e.g., $a$ is positive if $a \geq 0$), and the qualifier "strictly" will indicate the narrow sense (e.g., $a$ is strictly positive if $a > 0$). The set of all continuous real-valued functions on a topological space $X$ is denoted, as usual, $C(X)$, and its subset consisting of positive functions is denoted $C^+(X)$. We abbreviate expressions such as $C([0,T])$ and $C^+([0,T])$, where $T > 0$, $C([0,\infty))$, and $C^+([0,\infty))$ to $C[0,T]$, $C^+[0,T]$, $C[0,\infty)$, and $C^+[0,\infty)$, respectively, and let $C_a[0,T]$, $C_a^+[0,T]$, $C_a[0,\infty)$, and $C_a^+[0,\infty)$ stand for the subsets of these sets consisting of the functions $f$ satisfying $f(0) = a$, for a given constant $a$.

Let $\mathbb{N} := \{1, 2, \ldots\}$ be the set of all strictly positive integers, and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ be the set of all positive integers.

As usual $a \wedge b$ stands for minimum of $a$ and $b$ and $a \vee b$ for their maximum. In this paper, the operators $\wedge$ and $\vee$ have higher precedence than the arithmetic operators: e.g., $a + b \wedge c$ means $a + (b \wedge c)$. Other conventions of this kind are that:

- Cartesian product $\times$ has higher precedence than union $\cup$; so that, e.g., $A \cup \{1\} \times [0,\infty)$ means $A \cup (\{1\} \times [0,\infty))$;
- implicit multiplication (not using a multiplication sign such as $\times$ or $\cdot$) has higher precedence than division; so that, e.g., $S/NL$ means $S/(NL)$.

In our informal discussions we will use symbols $\approx$ for approximate equality and $\lesssim$ and $\gtrsim$ for approximate inequalities.

In this paper we consider a finite time interval $[0,T]$ where $T \in (0,\infty)$; without loss of generality we set $T := 1$.

# 2 The main result

The *sample space* used in this paper, $\Omega := C_1^+[0,1]$, is the set of all positive continuous functions $\omega : [0,1] \to [0,\infty)$ such that $\omega(0) = 1$. Intuitively, the functions in $\Omega$ are price paths of a financial security whose initial price serves as the unit for measuring its later prices.

We equip $\Omega$ with the usual $\sigma$-algebra $\mathcal{F}$, i.e., the smallest $\sigma$-algebra making all functions $\omega \in \Omega \mapsto \omega(t)$, $t \in [0,1]$, measurable. A *process* (more fully, an *adapted process*) $\mathfrak{S}$ is a family of extended random variables $\mathfrak{S}_t : \Omega \to [-\infty,\infty]$, $t \in [0,\infty)$, such that, for all $\omega, \omega' \in \Omega$ and all $t \in [0,\infty)$,

$$\omega|_{[0,t]} = \omega'|_{[0,t]} \Longrightarrow \mathfrak{S}_t(\omega) = \mathfrak{S}_t(\omega');$$

its *sample paths* are the functions $t \in [0,1] \mapsto \mathfrak{S}_t(\omega)$. A *stopping time* is an extended random variable $\tau : \Omega \to [0,\infty]$ such that, for all $\omega, \omega' \in \Omega$,

$$\omega|_{[0,\tau(\omega)\wedge 1]} = \omega'|_{[0,\tau(\omega)\wedge 1]} \Longrightarrow \tau(\omega) = \tau(\omega'),$$

where $\omega|_A$ stands for the restriction of $\omega$ to $A \subseteq [0,1]$. For any stopping time $\tau$, the $\sigma$-algebra $\mathcal{F}_\tau$ is defined as the family of all events $E \in \mathcal{F}$ such that, for all $\omega, \omega' \in \Omega$,

$$\bigl(\omega|_{[0,\tau(\omega)\wedge 1]} = \omega'|_{[0,\tau(\omega)\wedge 1]}, \omega \in E\bigr) \Longrightarrow \omega' \in E. \tag{1}$$

Therefore, a random variable $X$ is $\mathcal{F}_\tau$-measurable if and only if, for all $\omega, \omega' \in \Omega$,

$$\omega|_{[0,\tau(\omega)\wedge 1]} = \omega'|_{[0,\tau(\omega)\wedge 1]} \Longrightarrow X(\omega) = X(\omega').$$

**Remark 1.** Our definitions (convenient for the purposes of this paper) are equivalent to the standard ones by Galmarino's test ([3], IV.100).

First we define game-theoretic probability and expectation, partly following Perkowksi and Prömel [8, 7, 1] (this is a "broad" definition making our task easier; the older "narrow" definition of [11] is much more conservative and might require stronger assumptions for our main result to hold true; another broad definition was given in [12]). A *simple trading strategy* $G$ consists of an increasing sequence of stopping times $\tau_1 \le \tau_2 \le \cdots$ (we may assume, without loss of generality, $\tau_n \in [0,1] \cup \{\infty\}$ and $\tau_n < \tau_{n+1}$ unless $\tau_n = \infty$) and, for each $n = 1, 2, \ldots$, a bounded $\mathcal{F}_{\tau_n}$-measurable function $h_n$. It is required that, for each $\omega \in \Omega$, $\lim_{n\to\infty} \tau_n(\omega) = \infty$. To such $G$ and an *initial capital* $c \in \mathbb{R}$ corresponds the *simple capital process*

$$\mathcal{K}_t^{G,c}(\omega) := c + \sum_{n=1}^{\infty} h_n(\omega)\bigl(\omega(\tau_{n+1}(\omega) \wedge t) - \omega(\tau_n(\omega) \wedge t)\bigr), \quad t \in [0,\infty); \tag{2}$$

the value $h_n(\omega)$ will be called the *bet* (or *bet on* $\omega$, or *stake*) at time $\tau_n$, and $\mathcal{K}_t^{G,c}(\omega)$ will be called the *capital* at time $t$. For $c \ge 0$, let $\mathcal{C}_c$ be the class of positive functionals of the form $\mathcal{K}_1^{G,c}$, $G$ ranging over simple trading strategies; intuitively, these are the functionals that can be hedged with initial capital $c$ by a simple strategy that does not risk bankruptcy (notice that $\forall\omega : \mathcal{K}_1^{G,c}(\omega) \ge 0$ implies $\forall\omega\, \forall t : \mathcal{K}_t^{G,c}(\omega) \ge 0$).

A class $\mathcal{C}$ of functionals $F : \Omega \to [0,\infty]$ is $\liminf$*-closed* if $F \in \mathcal{C}$ whenever there is a sequence $F_1, F_2, \ldots$ of functionals in $\mathcal{C}$ such that

$$\forall\omega \in \Omega : F(\omega) \le \liminf_{n\to\infty} F_n(\omega). \tag{3}$$

The intuition is that if $F_1, F_2, \ldots$ can be superhedged, so can $F$ in the limit. It is clear that for each class $\mathcal{C}$ of functionals there is a smallest $\liminf$-closed class, denoted $\overline{\mathcal{C}}$, containing $\mathcal{C}$.

The *upper game-theoretic expectation* of a functional $F : \Omega \to [0,\infty]$ is defined to be

$$\overline{\mathbb{E}}^{\mathrm{g}}(F) := \inf\left\{c \mid F \in \overline{\mathcal{C}_c}\right\}. \tag{4}$$

where $\mathcal{C}_c$ is as defined above. The *upper game-theoretic probability* of $E \subseteq \Omega$ is $\overline{\mathbb{P}}^{\mathrm{g}}(E) := \overline{\mathbb{E}}^{\mathrm{g}}(\mathbf{1}_E)$, $\mathbf{1}_E$ being the indicator function of $E$.

The *upper measure-theoretic expectation* of $F$ is defined to be

$$\overline{\mathbb{E}}^{\mathrm{m}}(F) := \sup_P \int F dP,$$

where $P$ ranges over all *martingale measures*, i.e., probability measures on $\Omega$ under which the process $X_t(\omega) := \omega(t)$ is a martingale, and $\overline{\int}$ stands for upper integral. The *upper measure-theoretic probability* of $E \subseteq \Omega$ is $\overline{\mathbb{P}}^{\mathrm{m}}(E) := \overline{\mathbb{E}}^{\mathrm{m}}(\mathbf{1}_E)$.

Now we can state our main result, Theorem 2, in which "lower semicontinuous" refers to the standard topology on $\Omega$ generated by the usual uniform metric

$$\rho_U(\omega, \omega') := \sup_{t \in [0,1]} |\omega(t) - \omega'(t)|. \tag{5}$$

**Theorem 2.** *For any lower semicontinuous functional* $F : \Omega \to [0, \infty]$*,*

$$\overline{\mathbb{E}}^{\mathrm{g}}(F) = \overline{\mathbb{E}}^{\mathrm{m}}(F)$$

*(the inequality* $\geq$ *holding for all* $F : \Omega \to [0, \infty]$*).*

An earlier result of the same kind is the discrete-time Theorem 1 of [10].

# 3 Proof of Theorem 2

In this section we prove the coincidence of $\overline{\mathbb{E}}^{\mathrm{g}}$ and $\overline{\mathbb{E}}^{\mathrm{m}}$ on "simple" (lower semicontinuous in this version of the paper) positive functionals. We prove the inequality $\geq$ in Subsection 3.1 and the inequality $\leq$ in Subsection 3.2. Notice that we can ignore $\omega \in \Omega$ such that $0 = \omega(t) < \omega(s)$ for some $0 \leq t < s$.

On a few occasions we will use the following simple lemma.

**Lemma 3.** *The functions* $\overline{\mathbb{E}}^{\mathrm{g}}$ *and* $\overline{\mathbb{E}}^{\mathrm{m}}$ *are* $\sigma$*-subadditive: for any sequence of positive functionals* $F_1, F_2, \ldots$ *(taking values in* $[0, \infty]$*),*

$$\overline{\mathbb{E}}^{\mathrm{g}}\left(\sum_{n=1}^{\infty} F_n\right) \leq \sum_{n=1}^{\infty} \overline{\mathbb{E}}^{\mathrm{g}}(F_n), \tag{6}$$

$$\overline{\mathbb{E}}^{\mathrm{m}}\left(\sum_{n=1}^{\infty} F_n\right) \leq \sum_{n=1}^{\infty} \overline{\mathbb{E}}^{\mathrm{m}}(F_n). \tag{7}$$

*(And therefore, the set functions* $\overline{\mathbb{P}}^{\mathrm{g}}$ *and* $\overline{\mathbb{P}}^{\mathrm{m}}$ *are outer measures.)*

*Proof.* We can deduce (7) from the $\sigma$-subadditivity of $F \mapsto \overline{\int} F dP$: indeed, for each $\epsilon > 0$,

$$\begin{aligned}\overline{\mathbb{E}}^{\mathrm{m}}\left(\sum_{n=1}^{\infty} F_n\right) &= \sup_P \overline{\int} \sum_{n=1}^{\infty} F_n dP \leq \overline{\int} \sum_{n=1}^{\infty} F_n dP_0 + \epsilon \leq \sum_{n=1}^{\infty} \overline{\int} F_n dP_0 + \epsilon \\ &\leq \sum_{n=1}^{\infty} \sup_P \overline{\int} F_n dP + \epsilon = \sum_{n=1}^{\infty} \overline{\mathbb{E}}^{\mathrm{m}}(F_n) + \epsilon,\end{aligned}$$

where $P_0$ is a martingale measure.

As for (6), we start from a new definition of $\overline{\mathcal{C}_c}$. Define $\mathcal{C}_c^\alpha$ by transfinite induction over the countable ordinals $\alpha$ (see, e.g., [3], 0.8) as follows:

- $\mathcal{C}_c^0 := \mathcal{C}_c$;
- for $\alpha > 0$, $F \in \mathcal{C}_c^\alpha$ if and only if there exists a sequence $F_1, F_2, \ldots$ of functionals in $\mathcal{C}_c^{<\alpha} := \cup_{\beta<\alpha}\mathcal{C}_c^\beta$ such that (3) holds.

It is easy to check that $\overline{\mathcal{C}_c}$ is the union of the nested family $\mathcal{C}_c^\alpha$ over all countable ordinals $\alpha$.

First we prove finite subadditivity ((6) with $\infty$ replaced by a natural number), which will immediately follow from

$$\left(F_i \in \overline{\mathcal{C}_{c_i}},\ i = 1, \ldots, n\right) \Longrightarrow \left(\sum_{i=1}^n F_i \in \overline{\mathcal{C}_{\sum_{i=1}^n c_i}}\right).$$

It suffices to prove, for each countable ordinal $\alpha$,

$$\left(F_i \in \mathcal{C}_{c_i}^\alpha,\ i = 1, \ldots, n\right) \Longrightarrow \left(\sum_{i=1}^n F_i \in \mathcal{C}_{\sum_{i=1}^n c_i}^\alpha\right) \tag{8}$$

(this is the implication that we will actually need below). This is true for $\alpha = 0$ (by the definition of a simple trading strategy), so we fix a countable ordinal $\alpha > 0$ and assume that the statement holds for all ordinals below $\alpha$. Let us also assume the antecedent of (8). For each $i \in \{1, \ldots, n\}$ let $F_i^j \in \mathcal{C}_c^{<\alpha}$, $j = 1, 2, \ldots$, be a sequence such that

$$\forall \omega \in \Omega : F_i(\omega) \le \liminf_{j\to\infty} F_i^j(\omega).$$

For each $j$, the inductive assumption gives

$$\sum_{i=1}^n F_i^j \in \mathcal{C}_{\sum_{i=1}^n c_i}^{<\alpha}$$

(since there are finitely many $i$, there is $\beta = \beta_j < \alpha$ such that $F_i^j \in \mathcal{C}_{c_i}^\beta$ for all $i \in \{1, \ldots, n\}$). By the definition of $\mathcal{C}^\alpha$,

$$\liminf_{j\to\infty} \sum_{i=1}^n F_i^j \in \mathcal{C}_{\sum_{i=1}^n c_i}^\alpha,$$

which implies, by the Fatou lemma,

$$\sum_{i=1}^n \liminf_{j\to\infty} F_i^j \in \mathcal{C}_{\sum_{i=1}^n c_i}^\alpha,$$

which in turn implies

$$\sum_{i=1}^{n} F_i \in \mathcal{C}^{\alpha}_{\sum_{i=1}^{n} c_i}.$$

The countable subadditivity (6) now follows immediately from Lemma 5 below:

$$\overline{\mathbb{E}}^{\mathrm{g}}\left(\sum_{n=1}^{\infty} F_n\right) = \overline{\mathbb{E}}^{\mathrm{g}}\left(\liminf_{N\to\infty}\sum_{n=1}^{N} F_n\right) \le \liminf_{N\to\infty} \overline{\mathbb{E}}^{\mathrm{g}}\left(\sum_{n=1}^{N} F_n\right) \le \liminf_{N\to\infty}\sum_{n=1}^{N} \overline{\mathbb{E}}^{\mathrm{g}}(F_n) = \sum_{n=1}^{\infty} \overline{\mathbb{E}}^{\mathrm{g}}(F_n). \qquad \square$$

**Remark 4.** The original "broad" definition of game-theoretic probability and expectation in [8] is given by (4) with $\mathcal{C}^1_c$ in place of $\overline{\mathcal{C}_c}$.

The following lemma (already used in the proof of Lemma 3 above) is the analogue of the Fatou lemma for the broad definition of game-theoretic probability.

**Lemma 5.** *For any sequence of positive functionals $F_1, F_2, \ldots$,*

$$\overline{\mathbb{E}}^{\mathrm{g}}\left(\liminf_{n\to\infty} F_n\right) \le \liminf_{n\to\infty} \overline{\mathbb{E}}^{\mathrm{g}}(F_n). \tag{9}$$

*Proof.* Let $c$ be the right-hand side of (9) and $\epsilon > 0$. There is a strictly increasing sequence $n_1 < n_2 < \cdots$ such that $\overline{\mathbb{E}}^{\mathrm{g}}(F_{n_i}) < c + \epsilon$ for all $i$. Since $F_{n_i} \in \overline{\mathcal{C}_{c+\epsilon}}$ for all $i$, we have $\liminf_{i\to\infty} F_{n_i} \in \overline{\mathcal{C}_{c+\epsilon}}$, which implies $\liminf_{n\to\infty} F_n \in \overline{\mathcal{C}_{c+\epsilon}}$, which in turn implies $\overline{\mathbb{E}}^{\mathrm{g}}(\liminf_{n\to\infty} F_n) \le c+\epsilon$. Since $\epsilon$ can be made arbitrarily small, this completes the proof. $\square$

### 3.1 Inequality $\ge$

The goal of this subsection is to prove

$$\overline{\mathbb{E}}^{\mathrm{m}}(F) \le \overline{\mathbb{E}}^{\mathrm{g}}(F) \tag{10}$$

for all functionals $F : \Omega \to [0, \infty]$ (we will not need the assumptions that $F$ is bounded or measurable).

First we will prove

$$\mathbb{E}_P\left(\mathcal{K}^{G,c}_1 - c\right) \le 0 \tag{11}$$

for all martingale measures $P$, where $G$ is a simple trading strategy whose stopping times and bets will be denoted $\tau_1, \tau_2, \ldots$ and $h_1, h_2, \ldots$, respectively, and $c$ is an initial capital. Fix such a $P$. By the Fatou lemma (applied to the partial sums in (2)), it suffices to prove (11) assuming that the sequence of stopping time is finite: $\tau_n = \infty$ for all $n > N$ for a given $N \in \mathbb{N}$ (which in turn implies that the bets $h_n$ are bounded in absolute value by a given constant).

For each $k = 1, 2, \ldots$, set $\tau_n^k := 2^{-k}\lceil 2^k \tau_n \rceil$ and let $\mathfrak{S}^k$ be the simple capital process corresponding to initial capital $\mathfrak{S}_0^k = c$, stopping times $\tau_n^k$, and bets $h_n$ (remember that our definition of a simple trading strategy allows $\tau_n = \tau_{n+1}$). It is easy to check that, for all $k$ and $n = 0, \ldots, 2^k - 1$,

$$\mathbb{E}_P \left( \mathfrak{S}^k_{(n+1)2^{-k}} - \mathfrak{S}^k_{n2^{-k}} \right) = 0; \tag{12}$$

indeed, the difference $\mathfrak{S}^k_{(n+1)2^{-k}} - \mathfrak{S}^k_{n2^{-k}}$ is the product of the bounded $\mathcal{F}_{n2^{-k}}$-measurable function

$$h := \sum_{i=1}^{N} h_i \mathbf{1}_{\{\tau_i^k = n2^{-k}, \tau_{i+1}^k > n2^{-k}\}}$$

and the martingale difference $\omega((n+1)2^{-k}) - \omega(n2^{-k})$, and so

$$\begin{aligned}
&\mathbb{E}_P \left( \mathfrak{S}^k_{(n+1)2^{-k}} - \mathfrak{S}^k_{n2^{-k}} \right) \\
&= \mathbb{E}_{P(d\omega)} \left( \mathbb{E}_{P(d\omega)} \left( h(\omega) \left( \omega((n+1)2^{-k}) - \omega(n2^{-k}) \right) \mid \mathcal{F}_{n2^{-k}} \right) \right) \\
&= \mathbb{E}_{P(d\omega)} \left( h(\omega)\, \mathbb{E}_{P(d\omega)} \left( \left( \omega((n+1)2^{-k}) - \omega(n2^{-k}) \right) \mid \mathcal{F}_{n2^{-k}} \right) \right) = 0.
\end{aligned}$$

Summing (12) over $n$ (of which there are finitely many),

$$\mathbb{E}_P \left( \mathfrak{S}_1^k - c \right) = 0,$$

which in turn implies, by the Fatou lemma, (11).

We will complete the proof of (10) by transfinite induction, as in Lemma 3. Rewrite (10) as $\overline{\mathbb{E}}^{\mathrm{m}}(F) \le c$ for all $F \in \overline{\mathcal{C}_c}$. Fix $c$ and $F \in \overline{\mathcal{C}_c}$. In the previous paragraph we checked that $\overline{\mathbb{E}}^{\mathrm{m}}(F) \le c$ if $F \in \mathcal{C}_c^0$. Therefore, it remains to prove, for a given countable ordinal $\alpha > 0$, that $\overline{\mathbb{E}}^{\mathrm{m}}(F) \le c$ assuming that $F \in \mathcal{C}_c^\alpha$ and that $\overline{\mathbb{E}}^{\mathrm{m}}(G) \le c$ for all $G \in \mathcal{C}_c^{<\alpha}$. Let $F_n \in \mathcal{C}_c^{<\alpha}$, $n = 1, 2, \ldots$, be a sequence of functionals such that $F \le \liminf_n F_n$. Suppose $\overline{\mathbb{E}}^{\mathrm{m}}(F) > c$ and find a martingale measure $P$ such that $c < \int F dP$. We get a contradiction by the Fatou lemma and the inductive assumption:

$$c < \int F dP \le \int \liminf_{n\to\infty} F_n dP \le \liminf_{n\to\infty} \int F_n dP \le \liminf_{n\to\infty} c = c.$$

## 3.2 Inequality $\le$

In this section we will prove that

$$\overline{\mathbb{E}}^{\mathrm{g}}(F) \le \overline{\mathbb{E}}^{\mathrm{m}}(F). \tag{13}$$

Since $\overline{\mathbb{E}}^{\mathrm{g}}(F)$ is defined as an infimum and $\overline{\mathbb{E}}^{\mathrm{m}}(F)$ as a supremum, it suffices to construct a martingale measure $P$ and a superhedging capital process for a given lower semicontinuous positive functional $F$ such that $\int F dP$ is close to (or greater than) the initial capital of the process.

### 3.2.1 Reductions I

The goal of this section is to show that, without loss of generality, we can assume that the functional $F$ is bounded and lower semicontinuous in a stronger sense.

For a general lower semicontinuous $F : \Omega \to [0, \infty]$ and $n \in \mathbb{N}$, set $F_n(\omega) := F(\omega) \wedge n$. Assuming $\overline{\mathbb{E}}^{\mathrm{g}}(F_n) \le \overline{\mathbb{E}}^{\mathrm{m}}(F_n)$ for all $n$, let us prove $\overline{\mathbb{E}}^{\mathrm{g}}(F) \le \overline{\mathbb{E}}^{\mathrm{m}}(F)$. Set $c := \overline{\mathbb{E}}^{\mathrm{m}}(F) + \epsilon$ for a small $\epsilon > 0$. Since

$$\overline{\mathbb{E}}^{\mathrm{g}}(F_n) \le \overline{\mathbb{E}}^{\mathrm{m}}(F_n) \le \overline{\mathbb{E}}^{\mathrm{m}}(F) < c,$$

we have $F_n \in \overline{\mathcal{C}_c}$. Since $\overline{\mathcal{C}_c}$ is lim inf-closed, we have

$$F = \liminf_{n \to \infty} F_n \in \overline{\mathcal{C}_c}$$

and, therefore, $\overline{\mathbb{E}}^{\mathrm{g}}(F) \le c$. Since $\epsilon$ can be arbitrarily small, this completes the proof of $\overline{\mathbb{E}}^{\mathrm{g}}(F) \le \overline{\mathbb{E}}^{\mathrm{m}}(F)$. Therefore, we can, and will, assume that $F$ is bounded above.

In the rest of this paper, instead of the uniform metric (5) we will consider the Hausdorff metric

$$\rho_H(\omega, \omega') := H(\bar{\omega}, \bar{\omega'}) := \sup_{(t,x) \in \bar{\omega}} \inf_{(t',x') \in \bar{\omega'}} \|(t - t', x - x')\| \\ \vee \sup_{(t',x') \in \bar{\omega'}} \inf_{(t,x) \in \bar{\omega}} \|(t - t', x - x')\|, \quad (14)$$

where $\|\cdot\| = \|\cdot\|_\infty$ stands for the $\ell_\infty$ norm $\|(a, b)\| := |a| \vee |b|$ in $\mathbb{R}^2$ and each element $\omega$ of $\Omega$ is mapped to the set $\bar{\omega} \subseteq [0, 1] \times [0, \infty)$ defined to be the union $\mathrm{graph}(\omega) \cup \{1\} \times [0, \infty)$ of the graph of $\omega$ and the ray $\{1\} \times [0, \infty)$.

**Remark 6.** Notice that the metrics $\rho_U$ and $\rho_H$ lead to different topologies: e.g., there is an unbounded sequence $\omega_n$ of elements of $\Omega$ such that $\omega_n \to 0$ in $\rho_H$. The $\ell_\infty$ norm (used in our definition of $\rho_H$) is, of course, equivalent to the Euclidean norm $\ell_2$, but sometimes it leads to slightly simpler formulas. An example of a functional $F : \Omega \to \mathbb{R}$ continuous in $\rho_H$ is $F(\omega) := F'(\omega|_{[0,1-\epsilon]})$, where $F'$ is a functional on $C_1^+[0, 1 - \epsilon]$ continuous in the uniform metric and $\epsilon \in (0, 1)$ is a strictly positive constant.

**Remark 7.** On the other hand, the topologies generated by the metrics $\rho_U$ and $\rho_H$ lead to the same Borel $\sigma$-algebra. Since the topology generated by $\rho_U$ is finer than the one generated by $\rho_H$, it suffices to check that every $\rho_U$-Borel set is a $\rho_H$-Borel set. Since the $\rho_U$-topology is separable, it suffices to check that every open ball in $\rho_U$ is a $\rho_H$-Borel set. This is easy; moreover, every open ball in $\rho_U$ is the intersection of a sequence of $\rho_H$-open sets.

Let us check that in Theorem 2 we can further assume that $F$ is lower semicontinuous in the Hausdorff topology on $\Omega$ (this observation develops the end of Remark 6). Suppose that Theorem 2 holds for all (bounded) positive functionals that are lower semicontinuous in the Hausdorff topology. It is clear that we can replace the sample space $\Omega$ by the sample space $\Omega^* := C_1^+[0,2]$; let us do so. Now let $F$ be a lower semicontinuous (in the usual uniform topology) positive functional on $\Omega$. Define

$$F^*(\omega) := F(\omega|_{[0,1]}), \quad \omega \in \Omega^*.$$

Then $F^*$ is lower semicontinuous in the Hausdorff metric on $\Omega^*$ (defined by (14) where the ray $\{1\} \times [0,\infty)$ in the definition of $\bar{\omega}$ is replaced by $\{2\} \times [0,\infty)$). Indeed, for any constant $c$, the set $\{F^* > c\}$ is open: if $\omega_n \to \omega$ in the Hausdorff metric on $C_1^+[0,2]$, then $\omega_n|_{[0,1]} \to \omega|_{[0,1]}$ in the usual topology (had $\omega_n|_{[0,1]}$ not converged to $\omega|_{[0,1]}$ in the usual topology, we could have found $\epsilon > 0$ and $t_n \in [0,1]$ such that $|\omega_n(t_n) - \omega(t_n)| > \epsilon$ for infinitely many $n$ and arrived at a contradiction by considering a limit point of those $t_n$), and so

$$\forall n : F^*(\omega_n) \le c \iff \forall n : F(\omega_n|_{[0,1]}) \le c \Longrightarrow F(\omega|_{[0,1]}) \le c \iff F^*(\omega) \le c.$$

Therefore, our assumption (the non-trivial part of Theorem 2 for the Hausdorff metric) gives

$$\overline{\mathbb{E}}^{\mathrm{g}}(F^*) \le \overline{\mathbb{E}}^{\mathrm{m}}(F^*),$$

and it suffices to prove $\overline{\mathbb{E}}^{\mathrm{g}}(F) \le \overline{\mathbb{E}}^{\mathrm{g}}(F^*)$ and $\overline{\mathbb{E}}^{\mathrm{m}}(F^*) \le \overline{\mathbb{E}}^{\mathrm{m}}(F)$.

First let us check that $\overline{\mathbb{E}}^{\mathrm{g}}(F) \le \overline{\mathbb{E}}^{\mathrm{g}}(F^*)$. This follows from the class $\overline{\mathcal{C}_c}$ dominating the class $\overline{\mathcal{C}_c^*}$ for all $c > 0$, where the class $\mathcal{C}_c$ is as defined above, the class $\mathcal{C}_c^*$ is the analogue of this class for the time interval $[0,2]$ rather than $[0,1]$, and a class $\mathcal{A}$ of functionals on $\Omega$ is said to *dominate* a class $\mathcal{B}$ of functionals on $\Omega^*$ if for any $G \in \mathcal{B}$ there exists $G' \in \mathcal{A}$ that *dominates* $G$ in the sense that, for any $\omega \in \Omega$,

$$G'(\omega) \ge \mathbb{E}_{W(d\xi)}(G(\omega\xi))$$

where $W$ is the Wiener measure on $C_0[0,1]$ and $\omega\xi : [0,2] \to [0,\infty)$ is the continuous combination of $\omega$ and $\xi$ defined as follows:

$$(\omega\xi)(t) := \begin{cases} \omega(t) & \text{if } t \in [0,1] \\ \omega(1) + \xi(t-1) & \text{if } t \in [1,\tau] \\ 0 & \text{if } t \in (\tau,1] \end{cases}$$

where

$$\tau := \inf\{t \in [1,2] \mid \omega(1) + \xi(t-1) = 0\}$$

with $\inf \emptyset := 2$. Indeed, assuming that $\overline{\mathcal{C}_c}$ dominates $\overline{\mathcal{C}_c^*}$ for all $c > 0$, we obtain

$$\overline{\mathbb{E}}^{\mathrm{g}}(F) = \inf\left\{c \mid F \in \overline{\mathcal{C}_c}\right\} \le \inf\left\{c \mid F^* \in \overline{\mathcal{C}_c^*}\right\} = \overline{\mathbb{E}}^{\mathrm{g}}(F^*),$$

where the inequality follows from the fact that, whenever $F^* \in \overline{\mathcal{C}_c^*}$, $F^*$ is dominated by some $G \in \overline{\mathcal{C}_c}$, which implies

$$\forall \omega \in \Omega : F(\omega) = \mathbb{E}_{W(d\xi)}(F^*(\omega\xi)) \le G(\omega),$$

which in turn implies $F \in \overline{\mathcal{C}_c}$. Therefore, it remains to prove that $\overline{\mathcal{C}_c} \sqsupseteq \overline{\mathcal{C}_c^*}$, where $\mathcal{A} \sqsupseteq \mathcal{B}$ stands for "$\mathcal{A}$ dominates $\mathcal{B}$". Let us fix $c > 0$. Our proof is by transfinite induction. The basis of induction $\mathcal{C}_c^0 \sqsupseteq \mathcal{C}_c^{*0}$ follows from the fact that $\mathcal{K}_2^{G,c}$ is always dominated by $\mathcal{K}_1^{G,c}$: indeed, for a fixed $\omega \in \Omega$, any simple capital process $\mathcal{K}_t^{G,c}(\omega\xi)$ over $\Omega^*$ is a supermartingale over $t \in [1,2]$ (see the beginning of the proof of Lemma 6.4 in [11]), where $\xi \sim W$, as above. It remains to prove that $\mathcal{C}_c^\alpha \sqsupseteq \mathcal{C}_c^{*\alpha}$ for each countable ordinal $\alpha > 0$ assuming $\mathcal{C}_c^\beta \sqsupseteq \mathcal{C}_c^{*\beta}$ for each $\beta < \alpha$. Let us make this assumption and let $G \in \mathcal{C}_c^{*\alpha}$. Find a sequence of functionals $G_n \in \mathcal{C}_c^{*<\alpha}$, $n = 1, 2, \ldots$, such that $G \le \liminf_{n\to\infty} G_n$. By the inductive assumption, for each $n$ there is $G'_n \in \mathcal{C}_c^{<\alpha}$ that dominates $G$. By the Fatou lemma we now have, for each $\omega \in \Omega$,

$$\mathbb{E}_{W(d\xi)}(G(\omega\xi)) \le \mathbb{E}_{W(d\xi)}(\liminf_{n\to\infty} G_n(\omega\xi)) \le \liminf_{n\to\infty} \mathbb{E}_{W(d\xi)}(G_n(\omega\xi)) \le \liminf_{n\to\infty} G'_n(\omega).$$

In other words, $G' := \liminf_{n\to\infty} G'_n \in \mathcal{C}_c^\alpha$ dominates $G$. This completes the proof of $\overline{\mathbb{E}}^{\mathrm{g}}(F) \le \overline{\mathbb{E}}^{\mathrm{g}}(F^*)$.

To check that $\overline{\mathbb{E}}^{\mathrm{m}}(F) \ge \overline{\mathbb{E}}^{\mathrm{m}}(F^*)$, i.e.,

$$\sup_P \int F dP \ge \sup_{P^*} \int F^* dP^*,$$

where $P$ ranges over the martingale measures on $\Omega$ and $P^*$ over the martingale measures on $\Omega^*$, it suffices to notice that for any $P^*$ we can take as $P$ the martingale measure defined by

$$P(E) := P^* \left( \left\{ \omega \in \Omega^* \mid \omega_{[0,1]} \in E \right\} \right)$$

for all measurable $E \subseteq \Omega$ (essentially, the restriction of $P^*$ to cylinder sets in $\Omega^*$).

From now on $F$ is assumed bounded and lower semicontinuous in the Hausdorff metric.

### 3.2.2 Reductions II

We further simplify the functional $F$ analogously to the series of reductions in [11], Section 10. We will modify the notation of [11] and write $\tilde{\omega}$ for $\mathrm{ntt}(\omega)$ (as defined in Section 5 of [11]) and $\phi_s$ for $\tau_s$ (also defined in Section 5 of [11]). Let the domain of $\tilde{\omega}$ be $[0, D(\omega)]$ or $[0, D(\omega))$ (it has this form for typical $\omega \in \Omega$).

Let $\Omega''$ be the family of all sets of the form $A \cup \{1\} \times [0, \infty)$ where $A \subseteq [0,1] \times [0, \infty)$ is a bounded closed set and $\Omega' \subseteq \Omega''$ be the set of all $A \in \Omega''$ satisfying

- each vertical cut $A^t := \{a \mid (t, a) \in A\} \subseteq [0, \infty)$ of $A$, where $t \in [0, 1)$, is non-empty and connected (i.e., is a closed interval);
- $A^0 \ni 1$ (and, automatically, $A^1 = [0, \infty)$).

**Lemma 8.** *The set $\Omega'$ is closed in $\Omega''$ (equipped with the Hausdorff metric).*

*Proof.* Let $A_n \to A$ for some $A_n \in \Omega'$, $n = 1, 2, \ldots$, and $A \in \Omega''$; our goal is to prove $A \in \Omega'$. Let $B > 0$ be such that $A \subseteq [0,1) \times [0,B] \cup \{1\} \times [0,\infty)$. First we check that each cut of $A$ is non-empty: indeed, suppose $A^t = \emptyset$ for $t \in (0,1)$ (the case $t \in \{0,1\}$ is trivial); since $[0,B]$ is compact, this implies $A^{t'} = \emptyset$ for all $t'$ in a neighbourhood of $t$; therefore, $(A')^t = \emptyset$ for all $A'$ in a Hausdorff neighbourhood of $A$. Now suppose there is $t \in [0,1)$ (the case $t = 0$ will be also covered by our argument) such that $A^t$ is not connected, say $A^t$ contains points both above and below $b \in (0,B) \setminus A^t$. Let $O$ be a connected open neighbourhood of $t$ and $\delta$ be a strictly positive constant such that, for all $s \in O$, $A^s$ contains points below $b-\delta$ or points above $b+\delta$ but does not contain points in $(b-\delta, b+\delta)$. Choose another connected open neighbourhood $O'$ of $t$ such that $\overline{O'} \subseteq O$. Let $O_n^+$ be the set of $s \in O'$ such that $A_n^s$ contains points above $b$ and $O_n^-$ be the set of $s \in O'$ such that $A_n^s$ contains points below $b$. Since, for sufficiently large $n$, $O_n^+$ and $O_n^-$ are disjoint sets that are closed in $O'$ (closed in $O'$ by the compactness of $[0,b]$ and $[b,B]$) and $O'$ is connected, either $O' = O_n^+$ or $O' = O_n^-$. This makes $A_n \to A$ impossible. The remaining condition, $A^0 \ni 1$, is obvious. □

Now it is easy to see that $\Omega'$ is the closure of $\bar{\Omega} := \{\bar{\omega} \mid \omega \in \Omega\}$ in $\Omega''$.

We extend the functional $F$ to the set $\Omega'$ by setting

$$F'(A) := \liminf_{\omega \twoheadrightarrow A} F(\omega),$$

where $\omega$ ranges over $\Omega$ and $\omega \twoheadrightarrow A$ is the convergence in the sense of the "one-sided Hausdorff metric" (defined in terms of $\ell_\infty$, as always in this paper): namely, the $\epsilon$-neighbourhood of $A$ is the set of $\omega \in \Omega$ such that

$$\sup_{(t,a) \in \operatorname{graph}(\omega)} \inf_{(t',a') \in A} |t - t'| \vee |a - a'| < \epsilon,$$

and $\liminf_{\omega \twoheadrightarrow A} F(\omega)$ is the limit of the infimum of $F$ over the $\epsilon$-neighbourhood of $A$ as $\epsilon \to 0$. Since no $\epsilon$-neighbourhood of $A \in \Omega'$ is empty for $\epsilon > 0$ (see Lemma 9 below), $F'$ takes values in $[0, \sup F]$. Notice that $F'$ is monotonic: $F'(A) \geq F'(B)$ when $A \subseteq B$.

**Lemma 9.** *Let $A \in \Omega'$ and $\epsilon > 0$. The $\epsilon$-neighbourhood of $A$ is not empty.*

*Proof.* Draw parallel vertical lines $t = i/n$, $i = 0, \ldots, n$, at regular intervals in the semi-infinite region $[0,1] \times [0,\infty)$ of the $(t,a)$-plane starting from $t = 0$ and ending at $t = 1$; the interval $1/n$ between the lines should be at most $\epsilon$: $1/n \leq \epsilon$. Similarly, draw parallel horizontal lines $a = i/n$, $i = 0, 1, \ldots$, at regular intervals in the same semi-infinite region $[0,1] \times [0,\infty)$ starting from $a = 0$. The region $[0,1] \times [0,\infty)$ will be split into squares of size at most $\epsilon \times \epsilon$; these squares can be partitioned into columns (each column consisting of squares with equal $t$-coordinates). Let us mark the squares whose intersection with $A$ is non-empty. It suffices to prove that in each column the marked squares form a contiguous array and that these arrays overlap for each pair of adjacent columns: indeed, in this case we will be able to travel in a continuous manner from the point $(0, 1)$ to the line $t = 1$ via marked squares.

Suppose there is an unmarked square such that there is a point $(t', a') \in A$ in a square below it (in the same column) and there is a point $(t'', a'') \in A$ in a square above it (in the same column). (Notice that this unmarked square cannot be in the right-most column, and so the column containing the unmarked square can be regarded as bounded since $A$ is bounded, apart from the line $t = 1$.) Suppose, for concreteness, $t' < t''$. All $t \in [t', t'']$ are now split into two disjoint closed sets: those for which there are $(t, a) \in A$ for $a$ above the unmarked square and those for which there are $(t, a) \in A$ for $a$ below the unmarked square. Since $[t', t'']$ is connected, one of those disjoint closed sets is empty, and we have arrived at a contradiction.

Now it is obvious that the arrays of marked squares overlap for each pair of adjacent columns: remember that the intersection of $A$ with the vertical line between the two columns is non-empty and connected. $\square$

Let us check that $F' : \Omega' \to [0, \infty)$ is lower semicontinuous and that $F'(\bar{\omega}) = F(\omega)$ for all $\omega \in \Omega$; the latter property can be written as $F'|_\Omega = F$, where $F'|_\Omega : \Omega \to [0, \infty)$ is defined by $F'|_\Omega(\omega) := F'(\bar{\omega})$. Indeed:

- Let $c := F'(A)$ and $\epsilon > 0$; we are required to prove that $F'(B) \geq c - \epsilon$ for all $B$ in an open Hausdorff ball around $A$. Let $\delta > 0$ be so small that $F(\omega) > c - \epsilon$ for all $\omega \in \Omega$ in the $\delta$-neighbourhood of $A$. Let $B$ be in the open $\delta/2$-ball around $A$ (in the sense of the Hausdorff metric). If $\omega$ is in the $\delta/2$-neighbourhood of $B$, then $\omega$ will be in the $\delta$-neighbourhood of $A$, and so $F(\omega) > c - \epsilon$. Therefore, for such $B$ we have $F'(B) \geq c - \epsilon$.

- Let $\omega \in \Omega$. We have $F'(\bar{\omega}) \leq F(\omega)$ since $\omega$ is in the $\epsilon$-neighbourhood of $\bar{\omega}$ for any $\epsilon > 0$. And the inequality $F'(\bar{\omega}) \geq F(\omega)$ follows from the lower semicontinuity of $F$ on $\Omega$ (in the metric $\rho_H$) and the fact that $\omega_n \twoheadrightarrow \bar{\omega}$ implies $\rho_H(\omega_n, \omega) \to 0$. To check the last statement, suppose that there is a subsequence of $\omega_n$ such that $\rho_H(\omega_n, \omega) \geq \epsilon$ for the subsequence, where $\epsilon > 0$; without loss of generality we can assume that for each element of the subsequence there is a point $(t_n, a_n) \in \text{graph}(\omega)$ such that $t_n \leq 1 - \epsilon$ and there are no points of $\text{graph}(\omega_n)$ in the square $[t_n - \epsilon, t_n + \epsilon] \times [a_n - \epsilon, a_n + \epsilon]$. Let $(t, a)$ be a limit point of $(t_n, a_n)$, which obviously exists and belongs to $\text{graph}(\omega)$. There is another subsequence of $\omega_n$ for which there are no points of $\text{graph}(\omega_n)$ in the square $[t - \epsilon/2, t + \epsilon/2] \times [a - \epsilon/2, a + \epsilon/2]$. This contradicts $\omega_n \twoheadrightarrow \bar{\omega}$: the distance from $(t, \omega_n(t))$ to any point of $\text{graph}(\omega)$ stays above a strictly positive constant as $n \to \infty$.

Let us now check that we can assume $F = F'|_\Omega$ where $F' : \Omega' \to [0, \infty)$ is continuous (in the Hausdorff metric). First suppose (13) holds for the restrictions to $\Omega$ of all continuous functions of the type $\Omega' \to [0, \infty)$, but we are given $F = F'|_\Omega$ for $F'$ that is only lower semicontinuous. Each lower semicontinuous function on a metric space (such as $\Omega'$ with the Hausdorff metric) is the limit of an increasing sequence of continuous functions (see, e.g., [4], 1.7.15(c)), so we can find an increasing sequence of continuous functionals $F_n \nearrow F'$ on $\Omega'$. Let $\epsilon > 0$. For each $n$, by assumption we have $F_n|_\Omega \in \overline{\mathcal{C}_c}$ where $c := \overline{\mathbb{E}}^{\mathrm{m}}(F)+\epsilon > \overline{\mathbb{E}}^{\mathrm{m}}(F_n|_\Omega)$. Since $\overline{\mathcal{C}_c}$ is lim inf-closed,

$$F = \liminf_{n\to\infty} F_n|_\Omega \in \overline{\mathcal{C}_c}.$$

Therefore, $\overline{\mathbb{E}}^{\mathrm{g}}(F) \le c = \overline{\mathbb{E}}^{\mathrm{m}}(F) + \epsilon$ and so, since $\epsilon$ can be arbitrarily small, $\overline{\mathbb{E}}^{\mathrm{g}}(F) \le \overline{\mathbb{E}}^{\mathrm{m}}(F)$.

Let us check that we can replace our new assumption of continuity by the assumption that $F$ depends on $\omega \in \Omega$ only via the values $\tilde{\omega}(iS/N)$ and $\phi_{iS/N}(\omega)$, $i = 1, \dots, N$ (remember that we are interested in the case $\tilde{\omega}(0) = \omega(0) = 1$), for some $S > 0$ and some $N \in \mathbb{N}$ (in particular, only via $\tilde{\omega}|_{[0,S]}$ and $\phi(\omega)|_{[0,S]}$). We ignore events of zero upper game-theoretic probability (such as the event that $\tilde{\omega}$ does not exist). Let $\epsilon > 0$ and let $S$ and $N$ be sufficiently large (we will explain later how large $S$ and $N$ should be for a given $\epsilon$). Let $A_1 \subseteq \Omega$ consist of all $\omega \in \Omega$ such that $D(\omega) > S$ ($D(\omega)$ is defined at the beginning of this subsubsection on p. 9). Take $S$ so large that the probability that a Brownian motion started from 1 at time 0 is positive over the time interval $[0, S]$ is less than $\epsilon$.

Let $\mathfrak{K} \subseteq C_1[0, S]$ be a compact set whose Wiener measure (the distribution of a Brownian motion $W^1$ on $C[0, S]$ starting from 1) is more than $1 - \epsilon$. Let $f$ be the optimal modulus of continuity for all $\psi \in \mathfrak{K}$:

$$f(\delta) := \sup_{\substack{(t_1,t_2)\in[0,S]^2:|t_1-t_2|\le\delta,\\ \psi\in\mathfrak{K}}} |\psi(t_1) - \psi(t_2)|, \quad \delta > 0;$$

$f$ is an increasing function, $f(a + b) \le f(a) + f(b)$ for all $a, b \in [0, \infty)$, and we know that $\lim_{\delta\to 0} f(\delta) = 0$ (cf. the Arzelà–Ascoli theorem). Extend $\mathfrak{K}$ by including in it all $\omega \in C_1[0, S]$ with $f$ as a modulus of continuity; $\mathfrak{K}$ will stay compact with $W^1(\mathfrak{K}) > 1 - \epsilon$. Let $A_2 := \{\omega \in \Omega \mid \tilde{\omega}|_{[0,S]} \notin \mathfrak{K}\}$, where $\tilde{\omega}|_{[0,S]}(t) := \tilde{\omega}(D(\omega))$ for $t$ such that $D(\omega) \le t \le S$.

Set $B := 1 + f(S)$; notice that $\sup \omega \le B$ for all $\omega \in \Omega \setminus (A_1 \cup A_2)$.

Define $D_N^{S,f} \subseteq [0, B]^N \times [0, 1]^N$ to be the set of all sequences

$$(x_1, \dots, x_N; v_1, \dots, v_N) \in [0, B]^N \times [0, 1]^N$$

satisfying

$$\begin{cases} v_0 := 0 \le v_1 \le \cdots \le v_N \le v_{N+1} := 1, \\ |x_j - x_i| \le f((j - i)S/N) \text{ for all } i, j \in \{0, \dots, N\} \text{ such that } i < j, \end{cases} \tag{15}$$

where $x_0 := 1$ (notice that we do not require $v_i < v_{i+1}$ when $v_{i+1} < 1$, in order to make the set (15) closed). Define a function $U_N^{S,f} : D_N^{S,f} \to [0, \sup F]$ by

$$U_N^{S,f}(x_1, \dots, x_N; v_1, \dots, v_N) := F'\left(A_N^{S,f}(x_1, \dots, x_N; v_1, \dots, v_N)\right), \tag{16}$$

where $F'$ is the continuous function on $\Omega'$ defined earlier and the set $A := A_N^{S,f}(x_1, \dots, x_N; v_1, \dots, v_N) \in \Omega'$ is defined by the following conditions:

- for all $i \in \{0, \dots, N\}$ and $t \in (v_i, v_{i+1})$,

$$A^t = [x_i \wedge x_{i+1} - f(S/N), x_i \vee x_{i+1} + f(S/N)]$$

(with $x_i \wedge x_{i+1} = x_i \vee x_{i+1} := x_N$ when $i = N$);

- for all $i, j \in \{0, \dots, N\}$ such that $i < j$ and $v_i < t := v_{i+1} = v_{i+2} = \dots = v_j < v_{j+1}$,

$$A^t = \left[ \bigwedge_{k=i}^{j+1} x_k - f(S/N), \bigvee_{k=i}^{j+1} x_k + f(S/N) \right];$$

- $A^0 = [1 \wedge x_1 - f(S/N), 1 \vee x_1 + f(S/N)]$;
- $A^1 = [0, \infty)$.

Therefore, $A$ consists of a sequence of horizontal slabs of width at least $2f(S/N)$ separated by vertical lines. This set contains $\{1\} \times [0, \infty)$ and, for all $i = 0, \dots, N$, also contains $(v_i, x_i)$.

The metric on $D_N^{S,f}$ is defined by

$$\rho\left((x_1, \dots, x_N; v_1, \dots, v_N), (x'_1, \dots, x'_N; v'_1, \dots, v'_N)\right) := \bigvee_{j=1}^{N} \rho_H\left((v_j, x_j), (v'_j, x'_j)\right), \quad (17)$$

where the metric $\rho_H$ on $[0, 1] \times [0, \infty)$ is defined by

$$\begin{aligned} \rho_H\left((v, x), (v', x')\right) &:= H\left(\{(v, x)\} \cup \{1\} \times [0, \infty), \{(v', x')\} \cup \{1\} \times [0, \infty)\right) \\ &= (|v - v'| \vee |x - x'|) \wedge (1 - v \wedge v'), \end{aligned} \quad (18)$$

$H$ standing for the Hausdorff metric defined in terms of the $\ell_\infty$ metric on $[0, 1] \times [0, \infty)$, as before.

**Lemma 10.** *Each function $U_N^{S,f}$ is continuous on $D_N^{S,f}$ under our definition* (16) *and the metric* (17).

*Proof.* Fix some $(x_1, \dots, x_N; v_1, \dots, v_N) \in D_N^{S,f}$. Let $(x_1^n, \dots, x_N^n; v_1^n, \dots, v_N^n) \in D_N^{S,f}$ for $n = 1, 2, \dots$ and $(x_1^n, \dots, x_N^n; v_1^n, \dots, v_N^n) \to (x_1, \dots, x_N; v_1, \dots, v_N)$ in $\rho$ as $n \to \infty$. It is easy to see that, in the Hausdorff metric,

$$A_N^{S,f}(x_1^n, \dots, x_N^n, v_1^n, \dots, v_N^n) \to A_N^{S,f}(x_1, \dots, x_N; v_1, \dots, v_N)$$

as $n \to \infty$. This implies

$$U_N^{S,f}(x_1^n, \dots, x_N^n; v_1^n, \dots, v_N^n) \to U_N^{S,f}(x_1, \dots, x_N; v_1, \dots, v_N)$$

as $n \to \infty$ and completes the proof. □

Define a functional $F_N^{S,f} : \Omega \to [0, \sup F]$ by

$$F_N^{S,f}(\omega) = U_N^{S,f}\big(\omega(\phi_{S/N}(\omega) \wedge 1), \omega(\phi_{2S/N}(\omega) \wedge 1), \dots, \omega(\phi_S(\omega) \wedge 1); \\ \phi_{S/N}(\omega) \wedge 1, \phi_{2S/N}(\omega) \wedge 1, \dots, \phi_S(\omega) \wedge 1\big), \qquad \omega \in \Omega; \quad (19)$$

when the argument on the right-hand side is outside the domain $D_N^{S,f}$ of $U_N^{S,f}$, set $F_N^{S,f}(\omega) := \sup F$.

The following lemma lists the main properties of the sequence of functionals $F_N^{S,f}$, $N = 1, 2, \dots$, that we will need.

**Lemma 11.** *For all* $\omega \in \Omega \setminus (A_1 \cup A_2)$,

$$\forall N : F_N^{S,f}(\omega) \le F(\omega)$$

*and*

$$\liminf_{N\to\infty} F_N^{S,f}(\omega) \ge F(\omega). \quad (20)$$

*Proof.* Notice that $\omega \notin A_1$ implies $\phi_S(\omega) = 1$; therefore, $\omega \in \Omega \setminus (A_1 \cup A_2)$ implies

$$\bar\omega \subseteq A_N^{S,f}\big(\omega(\phi_{S/N}(\omega) \wedge 1), \omega(\phi_{2S/N}(\omega) \wedge 1), \dots, \omega(\phi_S(\omega) \wedge 1); \\ \phi_{S/N}(\omega) \wedge 1, \phi_{2S/N}(\omega) \wedge 1, \dots, \phi_S(\omega) \wedge 1\big),$$

which immediately implies $F_N^{S,f}(\omega) \le F(\omega)$. Since for $\omega \in \Omega \setminus (A_1 \cup A_2)$ the Hausdorff distance between

$$A_N^{S,f}\big(\omega(\phi_{S/N}(\omega) \wedge 1), \omega(\phi_{2S/N}(\omega) \wedge 1), \dots, \omega(\phi_S(\omega) \wedge 1); \\ \phi_{S/N}(\omega) \wedge 1, \phi_{2S/N}(\omega) \wedge 1, \dots, \phi_S(\omega) \wedge 1\big)$$

and $\bar\omega$ tends to 0 as $N \to \infty$, we also have (20). □

Let us extend $U_N^{S,f}$ to the whole of

$$\big\{(x_1, \dots, x_N; v_1, \dots, v_N) \in [0, B]^N \times [0, 1]^N \mid v_1 \le \dots \le v_N\big\}$$

obtaining a continuous function $\tilde U_N$ taking values in $[0, \sup F]$; this is possible by the Tietze–Urysohn theorem (see, e.g., [4], 2.1.8). Since the domain of the function $\tilde U_N$ is compact (in the usual topology, let alone in the topology generated by $\rho$), this function is uniformly continuous. Finally, extend $\tilde U_N$ to the whole of

$$D_N := \big\{(x_1, \dots, x_N; v_1, \dots, v_N) \in [0, \infty)^N \times [0, 1]^N \mid v_1 \le \dots \le v_N\big\}$$

by

$$U_N(x_1, \dots, x_N; v_1, \dots, v_N) := \tilde U_N(x_1 \wedge B, \dots, x_N \wedge B; v_1, \dots, v_N).$$

The function $U_N$ inherits the uniform continuity of $\tilde{U}_N$.

Analogously to (19), define a functional $F_N$ by

$$F_N(\omega) = U_N\big(\omega(\phi_{S/N}(\omega)\wedge 1), \omega(\phi_{2S/N}(\omega)\wedge 1), \dots, \omega(\phi_S(\omega)\wedge 1); \\ \phi_{S/N}(\omega)\wedge 1, \phi_{2S/N}(\omega)\wedge 1, \dots, \phi_S(\omega)\wedge 1\big); \quad (21)$$

by the definition of $U_N$, $F_N(\omega) = F_N^{S,f}(\omega)$ when $\omega \in \Omega \setminus (A_1 \cup A_2)$.

Our task is now reduced to proving $\overline{\mathbb{E}}^{\mathrm{g}}(F_N) \le \overline{\mathbb{E}}^{\mathrm{m}}(F_N)$. To demonstrate this, we first notice that

$$\overline{\mathbb{P}}^{\mathrm{g}}(A_1) \le \epsilon, \qquad \overline{\mathbb{P}}^{\mathrm{g}}(A_2) \le \epsilon, \quad (22)$$

$$\overline{\mathbb{P}}^{\mathrm{m}}(A_1) \le \overline{\mathbb{P}}^{\mathrm{g}}(A_1) \le \epsilon, \qquad \overline{\mathbb{P}}^{\mathrm{m}}(A_2) \le \overline{\mathbb{P}}^{\mathrm{g}}(A_2) \le \epsilon; \quad (23)$$

indeed, (22) follows from Theorem 3.1 of [11] and the time-superinvariance of the sets

$$\{\omega \in C_1[0,\infty) \mid \tilde{\omega} \text{ is defined and positive over } [0,S]\}$$

and

$$\left\{\omega \in C_1[0,\infty) \mid \tilde{\omega} \text{ is defined over } [0,S] \text{ and } \tilde{\omega}|_{[0,S]} \notin \mathfrak{K}\right\},$$

and (23) follows from $\overline{\mathbb{P}}^{\mathrm{m}} \le \overline{\mathbb{P}}^{\mathrm{g}}$, established in the previous subsection: see (10). In combination with Lemmas 3, 5, 11, and the assumption $\overline{\mathbb{E}}^{\mathrm{g}}(F_N) \le \overline{\mathbb{E}}^{\mathrm{m}}(F_N)$, for all $N$, this implies

$$\begin{aligned}\overline{\mathbb{E}}^{\mathrm{g}}(F) &\le \overline{\mathbb{E}}^{\mathrm{g}}\left(\liminf_{N\to\infty} F_N^{S,f}\right) + 2C\epsilon \le \liminf_{N\to\infty} \overline{\mathbb{E}}^{\mathrm{g}}(F_N^{S,f}) + 2C\epsilon \\ &\le \liminf_{N\to\infty} \overline{\mathbb{E}}^{\mathrm{g}}(F_N) + 4C\epsilon \le \liminf_{N\to\infty} \overline{\mathbb{E}}^{\mathrm{m}}(F_N) + 4C\epsilon \\ &\le \liminf_{N\to\infty} \overline{\mathbb{E}}^{\mathrm{m}}(F_N^{S,f}) + 6C\epsilon \le \overline{\mathbb{E}}^{\mathrm{m}}(F) + 8C\epsilon\end{aligned}$$

for $C := \sup F$. Since $\epsilon$ can be arbitrarily small, this achieves our goal.

### 3.2.3 Setting intermediate goals

Let us fix $S$ and $N$; our goal is to prove $\overline{\mathbb{E}}^{\mathrm{g}}(F_N) \le \overline{\mathbb{E}}^{\mathrm{m}}(F_N)$. We will abbreviate $U_N$ to $U$.

We start the proof by defining functions

$$U_i^{\mathrm{e}} : D_i^{\mathrm{e}} \to [0,\infty), \quad i = 0,\dots,N,$$
$$U_i^{\mathrm{m}} : D_i^{\mathrm{m}} \to [0,\infty), \quad i = 0,\dots,N-1$$

(with "m" standing for "maximization" and "e" for "expectation") whose domains are

$$D_i^{\mathrm{e}} := \Big\{(x_1,v_1,\dots,x_i,v_i) \in ([0,\infty)\times[0,1])^i \mid \\ v_1 \le \dots \le v_i \text{ and } (x_j = x_{j+1} \text{ whenever } j < i \text{ and } v_j = 1)\Big\},$$

$$D_i^{\mathrm{m}} := \big\{(x_1,v_1,\dots,x_i,v_i,x_{i+1}) \in ([0,\infty)\times[0,1])^i \times [0,\infty) \mid$$

$$v_1 \le \cdots \le v_i \text{ and } (x_j = x_{j+1} \text{ whenever } j \le i \text{ and } v_j = 1)\big\}.$$

They will be defined by induction in $i$.

The basis of induction is

$$U^{\mathrm{e}}_N(x_1, v_1, \ldots, x_N, v_N) := U(x_1, \ldots, x_N; v_1, \ldots, v_N). \tag{24}$$

Given $U^{\mathrm{e}}_{i+1}$, where $i := N-1$, we define

$$U^{\mathrm{m}}_i(x_1, v_1, \ldots, x_i, v_i, x_{i+1}) := \sup_{v \in [v_i, 1]} U^{\mathrm{e}}_{i+1}(x_1, v_1, \ldots, x_i, v_i, x_{i+1}, v). \tag{25}$$

Given $U^{\mathrm{m}}_i$, where $i := N-1$, we next define

$$U^{\mathrm{e}}_i(x_1, v_1, \ldots, x_i, v_i) = \begin{cases} U^{\mathrm{m}}_i(x_1, v_1, \ldots, x_i, v_i, x_i) & \text{if } v_i = 1 \\ \mathbb{E}\, U^{\mathrm{m}}_i(x_1, v_1, \ldots, x_i, v_i, \xi) & \text{otherwise} \end{cases} \tag{26}$$

where $\xi \ge 0$ is the value at time $S/N$ of a linear Brownian motion that starts at $x_i$ at time 0 and is stopped when it hits level 0. Next use alternately (25) and (26) for

$$i = N-2, N-2; N-3, N-3; \ldots; 1, 1$$

to define inductively other $U^{\mathrm{m}}_i$ and $U^{\mathrm{e}}_i$. Finally, define

$$U^{\mathrm{m}}_0(x_1) := \sup_{v \in [0,1]} U^{\mathrm{e}}_1(x_1, v), \qquad U^{\mathrm{e}}_0 := \mathbb{E}\, U^{\mathrm{m}}_0(\xi)$$

where $\xi \ge 0$ is the value at time $S/N$ of a linear Brownian motion that starts at 1 at time 0 and is stopped when it hits level 0 (the last event being unlikely for a large $N$).

In this proof we will show that $U^{\mathrm{e}}_0$ is sandwiched between $\overline{\mathbb{E}}^{\mathrm{m}}(F_N)$ and $\overline{\mathbb{E}}^{\mathrm{g}}(F_N)$ as $\overline{\mathbb{E}}^{\mathrm{m}}(F_N) \ge U^{\mathrm{e}}_0 \ge \overline{\mathbb{E}}^{\mathrm{g}}(F_N)$, which will achieve our goal. But first we discuss some properties of regularity of the intermediate functions $U^{\mathrm{m}}_i$ and $U^{\mathrm{e}}_i$.

It is obvious that each of the functions $U^{\mathrm{e}}_i$ and $U^{\mathrm{m}}_i$ is bounded (by $\sup F$), and the following two lemmas imply that they are uniformly continuous. The metric on $D^{\mathrm{e}}_i$ is defined by

$$\rho^{\mathrm{e}}\left((x_1, v_1, \ldots, x_i, v_i), (x'_1, v'_1, \ldots, x'_i, v'_i)\right) := \bigvee_{j=1}^{i} \rho_H\left((v_j, x_j), (v'_j, x'_j)\right),$$

$\rho_H$ being defined in (18). The metric on $D^{\mathrm{m}}_i$ is defined by

$$\begin{aligned} &\rho^{\mathrm{m}}\left((x_1, v_1, \ldots, x_i, v_i, x_{i+1}), (x'_1, v'_1, \ldots, x'_i, v'_i, x'_{i+1})\right) \\ &\quad := \sup_{v \in [v_i \wedge v'_i, 1]} \rho^{\mathrm{e}}\Big((x_1, v_1, \ldots, x_i, v_i, x_{i+1}, v \vee v_i), \\ &\qquad\qquad (x'_1, v'_1, \ldots, x'_i, v'_i, x'_{i+1}, v \vee v'_i)\Big). \end{aligned}$$

**Lemma 12.** *If a function $U^{\mathrm{e}}_{i+1}$ on $D^{\mathrm{e}}_{i+1}$ is uniformly continuous, then the function $U^{\mathrm{m}}_i$ on $D^{\mathrm{m}}_i$ defined by (25) is also uniformly continuous (with the same modulus of continuity).*

*Proof.* Let $f$ be a modulus of continuity for $U^{\mathrm{e}}_{i+1}$ (in this paper we only consider increasing moduli of continuity). It suffices to prove that, for each $\delta > 0$,

$$\sup_{v\in[v_i,1]} U^{\mathrm{e}}_{i+1}(x_1,v_1,\dots,x_i,v_i,x_{i+1},v) \ge \sup_{v\in[v'_i,1]} U^{\mathrm{e}}_{i+1}(x'_1,v'_1,\dots,x'_i,v'_i,x'_{i+1},v) - f(\delta) \quad (27)$$

provided the $D^{\mathrm{m}}_i$ distance between $(x_1,v_1,\dots,x_{i+1})$ and $(x'_1,v'_1,\dots,x'_{i+1})$ does not exceed $\delta$. This follows from

$$\begin{aligned}\sup_{v\in[v_i,1]} U^{\mathrm{e}}_{i+1}(x_1,v_1,\dots,x_i,v_i,x_{i+1},v) &\ge U^{\mathrm{e}}_{i+1}(x_1,v_1,\dots,x_i,v_i,x_{i+1},v'\vee v_i)\\ &\ge U^{\mathrm{e}}_{i+1}(x'_1,v'_1,\dots,x'_i,v'_i,x'_{i+1},v') - f(\delta)\\ &= \sup_{v\in[v'_i,1]} U^{\mathrm{e}}_{i+1}(x'_1,v'_1,\dots,x'_i,v'_i,x'_{i+1},v) - f(\delta),\end{aligned}$$

where $v' \ge v'_i$ is the point at which the supremum on the right-hand side of (27) is attained. □

**Lemma 13.** *If a function $U^{\mathrm{m}}_i$ on $D^{\mathrm{m}}_i$ is bounded and uniformly continuous, then the function $U^{\mathrm{e}}_i$ on $D^{\mathrm{e}}_i$ defined by* (26) *is also uniformly continuous.*

*Proof.* Let $\delta > 0$ and $f$ be the optimal modulus of continuity for $U^{\mathrm{m}}_i$; to bound the optimal modulus of continuity for $U^{\mathrm{e}}_i$, we consider three possibilities for two points $E$ and $E'$ in $D^{\mathrm{e}}_i$ where $E = (x_1,v_1,\dots,x_i,v_i)$, $E' = (x'_1,v'_1,\dots,x'_i,v'_i)$, and $\rho^{\mathrm{e}}(E,E') \le \delta$.

- If $v_i = v'_i = 1$, the difference between $U^{\mathrm{e}}_i(E)$ and $U^{\mathrm{e}}_i(E')$ does not exceed $f(\delta)$.
- If $v_i < v'_i = 1$ or $v'_i < v_i = 1$, the difference between $U^{\mathrm{e}}_i(E)$ and $U^{\mathrm{e}}_i(E')$ also does not exceed $f(\delta)$. Indeed, suppose, for concreteness, that $v'_i = 1$. Then $v_i \ge 1-\delta$. By definition, $U^{\mathrm{e}}_i(E)$ is an average of $U^{\mathrm{m}}_i(E,x)$ over $x$, and $U^{\mathrm{e}}_i(E')$ coincides with $U^{\mathrm{m}}_i(E',x'_i)$. By the definition of the metric on $D^{\mathrm{m}}_i$, the $\rho^{\mathrm{m}}$ distance between $(E,x)$ and $(E',x'_i)$ is at most $\delta$, and so the difference between $U^{\mathrm{e}}_i(E)$ and $U^{\mathrm{e}}_i(E')$ does not exceed $f(\delta)$.
- If $v_i < 1$ and $v'_i < 1$, the difference between $U^{\mathrm{e}}_i(E)$ and $U^{\mathrm{e}}_i(E')$ does not exceed $2f(\delta)+C\delta\sqrt{N/S}$, where $C$ is an upper bound on $U^{\mathrm{m}}_i$. Let us check this. Our goal is to prove that

  $$|\mathbb{E}\,U^{\mathrm{m}}_i(E,\xi) - \mathbb{E}\,U^{\mathrm{m}}_i(E',\xi')| \le 2f(\delta) + C\delta\sqrt{N/S}$$

  where $\xi$ (resp. $\xi'$) is the value at time $S/N$ of a linear Brownian motion that starts at $x_i$ (resp. $x'_i$) at time 0 and is stopped when it hits level 0. It suffices to notice that

  $$|\mathbb{E}\,U^{\mathrm{m}}_i(E,\xi) - \mathbb{E}\,U^{\mathrm{m}}_i(E',\xi')|$$

$$\leq |\mathbb{E}\, U_i^{\mathrm{m}}(E,\xi) - \mathbb{E}\, U_i^{\mathrm{m}}(E,\xi')| + |\mathbb{E}\, U_i^{\mathrm{m}}(E,\xi') - \mathbb{E}\, U_i^{\mathrm{m}}(E',\xi')| \qquad (28)$$
$$\leq f(\delta) + C\delta/\sqrt{S/N} + f(\delta).$$

The upper bound $f(\delta) + C\delta/\sqrt{S/N}$ on the first addend in (28) follows from Lemma 14 below; we also used the uniform continuity of $U_i^{\mathrm{m}}(E,\cdot)$ and $U_i^{\mathrm{m}}(\cdot,x)$, where $x \in [0,\infty)$, with $f$ as modulus of continuity.

In all three cases the difference is bounded by $2f(\delta) + C\delta\sqrt{N/S}$. □

The following result was used in the proof of Lemma 13 above.

**Lemma 14.** *Suppose $a > 0$ and $u : [0,\infty) \to [0,C]$ is a bounded uniformly continuous function with $f$ as modulus of continuity. Then*

$$x \in [0,\infty) \mapsto \mathbb{E}\, u(W^x_{\tau\wedge a}),$$

*where $W^x$ is a Brownian motion started at $x$ and $\tau$ is the moment it hits level 0, is uniformly continuous with $\delta > 0 \mapsto f(\delta) + C\delta/\sqrt{a}$ as modulus of continuity.*

*Proof.* Consider points $x \in [0,\infty)$ and $x' \in (x, x+\delta]$, for some $\delta > 0$. Let us map each path of $W^{x'}_{\tau\wedge a}$ to the path of $W^x_{\tau\wedge a}$ obtained by subtracting $x'-x$ and stopping when level 0 is hit; we will refer to the latter as the path *corresponding* to the former. There are three kinds of paths of $W^{x'}_{\tau\wedge a}$:

- Those that never hit level $x'-x$ over the time interval $[0,a]$. The average of $u(W^{x'}_{\tau\wedge a}) = u(W^{x'}_a)$ over such paths and the average of $u(W^x_{\tau\wedge a}) = u(W^x_a)$ over the corresponding paths differ by at most $f(\delta)$.
- Those that hit level 0 over $[0,a]$. The average of $u(W^{x'}_{\tau\wedge a}) = u(0)$ over such paths and the average of $u(W^x_{\tau\wedge a}) = u(0)$ over the corresponding paths coincide.
- Those that hit level $x'-x$ but never hit level 0 over $[0,a]$. The probability of such paths is

$$2\Phi(-x/\sqrt{a}) - 2\Phi(-x'/\sqrt{a}) \leq 2\,\mathbb{P}(\xi \in [0,(x'-x)/\sqrt{a}])$$
$$< \frac{2}{\sqrt{2\pi}}(x'-x)/\sqrt{a} < \delta/\sqrt{a},$$

  where $\Phi$ is the standard normal distribution function, $\xi \sim \Phi$, and the factor of 2 comes from the reflection principle.

Therefore, the overall averages of $u(W^x_{\tau\wedge a})$ and $u(W^{x'}_{\tau\wedge a})$ differ by at most $f(\delta) + C\delta/\sqrt{a}$. □

### 3.2.4 Tackling measure-theoretic probability

First we prove an easy auxiliary statement ensuring the existence of measurable "choice functions".

**Lemma 15.** *Suppose $\{A_\theta \mid \theta \in \Theta\}$ is a countable cover of a measurable space $\Omega$ such that each $A_\theta$ is measurable. There is a measurable function $f : \Omega \to \Theta$ (with the discrete $\sigma$-algebra on $\Theta$) such that $\omega \in A_{f(\omega)}$ for all $\omega \in \Omega$.*

*Proof.* Assume, without loss of generality, $\Theta = \mathbb{N}$. Define

$$f(\omega) := \min\{\theta \mid \omega \in A_\theta\}.$$

Then, for each $\theta \in \mathbb{N}$, the set

$$\{\omega \mid f(\omega) \le \theta\} = A_1 \cup \cdots \cup A_\theta$$

is measurable. □

In this section we show that $\overline{\mathbb{E}}^{\mathrm{m}}(F_N) \ge U_0^{\mathrm{e}}$. We define a martingale measure $P$ by backward induction. For each $i = 0, \ldots, N-1$, let $V_{i+1}$ be a Borel function on $D_i^{\mathrm{m}}$ such that, for all $(x_1, v_1, \ldots, x_i, v_i, x_{i+1}) \in D_i^{\mathrm{m}}$ satisfying $v_i < 1$, it is true that

$$v_i < V_{i+1}(x_1, v_1, \ldots, x_i, v_i, x_{i+1}) < 1$$

and

$$U_{i+1}^{\mathrm{e}}\left(x_1, v_1, \ldots, x_i, v_i, x_{i+1}, V_{i+1}(x_1, v_1, \ldots, x_i, v_i, x_{i+1})\right) \\ \ge U_i^{\mathrm{m}}(x_1, v_1, \ldots, x_i, v_i, x_{i+1}) - \epsilon$$

(cf. (25)), where $\epsilon > 0$ is a small constant (further details will be added later). (Intuitively, $V_{i+1}$ outputs a $v > v_i$ at which the supremum of $U_{i+1}^{\mathrm{e}}(x_1, v_1, \ldots, x_{i+1}, v)$ is almost attained.) The existence of such $V_{i+1}$ follows from Lemma 15: indeed, for each rational $r \in (0, 1)$ the set

$$A_r := \bigl\{(x_1, v_1, \ldots, x_i, v_i, x_{i+1}) \in D_i^{\mathrm{m}} \mid r > v_i \text{ and} \\ U_{i+1}^{\mathrm{e}}(x_1, v_1, \ldots, x_i, v_i, x_{i+1}, r) \ge U_i^{\mathrm{m}}(x_1, v_1, \ldots, x_i, v_i, x_{i+1}) - \epsilon\bigr\}$$

is Borel (namely, intersection of open and closed), and the sets $A_r$ form a cover of $D_i^{\mathrm{m}}$. By the uniform continuity of $U_{i+1}^{\mathrm{e}}$ and $U_i^{\mathrm{m}}$, there is $\delta > 0$ such that, for all $i$ (remember that there are finitely many $i$) and for all $x_1$, $v_1$,…, $x_i$, $v_i$, $x_{i+1}$, and $x'_{i+1}$,

$$\left|x'_{i+1} - x_{i+1}\right| < \delta \\ \Longrightarrow U_{i+1}^{\mathrm{e}}\left(x_1, v_1, \ldots, x_i, v_i, x_{i+1}, V_{i+1}(x_1, v_1, \ldots, x_i, v_i, x'_{i+1})\right) \\ \ge U_i^{\mathrm{m}}(x_1, v_1, \ldots, x_i, v_i, x'_{i+1}) - 2\epsilon. \quad (29)$$

Next choose Borel $V_i^*$ such that, for $v_i < 1$,

$$V_{i+1}(x_1, v_1, \ldots, x_i, v_i, \xi) > V_i^*(x_1, v_1, \ldots, x_i, v_i) > v_i \quad (30)$$

with probability (over $\xi$ only) at least $1-\epsilon$ when $\xi$ is the value taken at time $S/N$ by a linear Brownian motion started from $x_i$ at time 0 and stopped when it hits level 0. (The existence of $V_i^*$ also follows from Lemma 15.) Let $\Delta \in (0, S/N)$ be such that

$$\sup_{t\in[0,\Delta]} |W_t| < \delta \tag{31}$$

with a probability at least $1-\epsilon$, where $W$ is a standard Brownian motion.

By a *scaled Brownian motion* we will mean a process of the type $W_{ct}$ where $W$ is a Brownian motion and $c > 0$ (equivalently, a process of the type $cW_t$ where $W$ is a Brownian motion and $c > 0$). Define a probability measure $P$ on $\Omega$ as the distribution of $\omega \in \Omega$ generated as follows. For $i = 0, 1, \dots, N-1$:

- Start a scaled Brownian motion $W^i$ (independent of what has happened before if $i > 0$) from $x_i$ (with $x_0 := 1$) at time $v_i$ (with $v_0 := 0$) such that its quadratic variation over $[v_i, v_i^*]$ is $S/N - \Delta$, where

  $$v_i^* := V_i^*(x_1, v_1, \dots, x_i, v_i) < 1.$$

  Define

  $$\omega|_{[v_i,v_i^*]} := W^{\circ,i}|_{[v_i,v_i^*]}$$

  where $W^{\circ,i}$ is $W^i$ stopped when it hits level 0. If $\omega(v_i^*) = 0$, the random process of generating $\omega$ is complete; set $\omega|_{[v_i^*,1]} := 0$, $v_{i+1}^* = \dots = v_{N-1}^* := 1$, and $v_{i+1} = \dots = v_N := 1$, and then stop.

- Set

  $$v_{i+1} := \begin{cases} V_{i+1}(x_1, v_1, \dots, x_i, v_i, \omega(v_i^*)) & \text{if } V_{i+1}(x_1, v_1, \dots, x_i, v_i, \omega(v_i^*)) > v_i^* \\ 1 & \text{otherwise.} \end{cases}$$

  Start another independent Brownian motion $\bar{W}^i$ from $\omega(v_i^*)$ at time $v_i^*$ such that its quadratic variation over $[v_i^*, v_{i+1}]$ is $\Delta$. Define

  $$\omega|_{[v_i^*,v_{i+1}]} := \bar{W}^{\circ,i}|_{[v_i^*,v_{i+1}]}$$

  where $\bar{W}^{\circ,i}$ is $\bar{W}^i$ stopped when it hits level 0. If $\omega(v_{i+1}) = 0$ or $v_{i+1} = 1$ (or both), the random process of generating $\omega$ is complete; set $\omega|_{[v_{i+1},1]} := 0$ if $v_{i+1} < 1$, set $v_{i+1}^* = \dots = v_{N-1}^* := 1$ and $v_{i+2} = \dots = v_N := 1$, and then stop.

- Set $x_{i+1} := \omega(v_{i+1})$; notice that $v_{i+1} < 1$.

If the procedure was not stopped, and so $v_N < 1$, define $\omega|_{[v_N,1]}$ to be the constant $x_N = \omega(v_N)$.

Let us now check that $\mathbb{E}_P(F_N) \geq U_0^{\mathrm{e}}$. More precisely, we will show by induction in $i$ that, for $i = N, \dots, 0$,

$$\mathbb{E}_P(F_N \mid \mathcal{F}_{\tilde{v}_i}) \geq U_i^{\mathrm{e}}(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i) - (N-i)(3C+3)\epsilon \quad \text{a.s.,} \tag{32}$$

and that, for $i = N-1, \dots, 0$,

$$\mathbb{E}_P(F_N \mid \mathcal{F}_{\tilde{v}_i^*}) \ge U_i^{\mathrm{m}}(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i, \omega(\tilde{v}_i^*)) \\ - (N-i)(3C+3)\epsilon + (C+1)\epsilon \quad \text{a.s.}, \quad (33)$$

where: $C := \sup U$; $\tilde{x}_j$ are $x_j$ (as defined in the definition of $P$) considered as function of $\omega$ (it is clear that $x_j$ can be restored given $\omega$ $P$-almost surely); similarly, $\tilde{v}_j$ and $\tilde{v}_j^*$ are $v_j$ and $v_j^*$ considered as functions of $\omega$; $\mathcal{F}_{\tilde{v}_i}$ and $\mathcal{F}_{\tilde{v}_i^*}$ are the usual $\sigma$-algebras on $\Omega$ defined as in (1) for the stopping times $\tilde{v}_i$ and $\tilde{v}_i^*$. Since, $\epsilon$ can be arbitrarily small, (32) with $i = 0$ will achieve our goal.

For $i = N$, (32) holds almost surely as $U_i^{\mathrm{e}} := U := U_N$ and $F_N$ is defined by (21).

Assuming (32) with $i+1$ in place of $i$, $i < N$, let us deduce (33): concentrating on the non-trivial case $\tilde{v}_i < 1$,

$$\begin{aligned}
&\mathbb{E}_P(F_N \mid \mathcal{F}_{\tilde{v}_i^*}) = \mathbb{E}_P\Big(\mathbb{E}_P(F_N \mid \mathcal{F}_{\tilde{v}_{i+1}}) \mid \mathcal{F}_{\tilde{v}_i^*}\Big) \\
&\ge \mathbb{E}_P\Big(U_{i+1}^{\mathrm{e}}(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i, \tilde{x}_{i+1}, \tilde{v}_{i+1}) \mid \mathcal{F}_{\tilde{v}_i^*}\Big) - (N-i-1)(3C+3)\epsilon \\
&\ge U_i^{\mathrm{m}}(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i, \omega(\tilde{v}_i^*)) - (N-i-1)(3C+3)\epsilon - (2C+2)\epsilon \\
&= U_i^{\mathrm{m}}(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i, \omega(\tilde{v}_i^*)) - (N-i)(3C+3)\epsilon + (C+1)\epsilon \quad \text{a.s.},
\end{aligned}$$

where the second inequality follows from the fact that

$$U_{i+1}^{\mathrm{e}}(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i, \tilde{x}_{i+1}, \tilde{v}_{i+1}) \ge U_i^{\mathrm{m}}(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i, \omega(\tilde{v}_i^*)) - 2\epsilon$$

with $\mathcal{F}_{\tilde{v}_i^*}$-conditional probability at least $1 - 2\epsilon$ a.s. This fact in turn follows from (30) and (31) each holding with probability at least $1 - \epsilon$ (and so the conjunction of $|\omega(\tilde{v}_{i+1}) - \omega(\tilde{v}_i^*)| < \delta$ and $\tilde{v}_{i+1} < 1$ holding with $\mathcal{F}_{\tilde{v}_i^*}$-conditional probability at least $1 - 2\epsilon$ a.s.) combined with an application of (29).

Assuming (33) let us deduce (32): again concentrating on the case $\tilde{v}_i < 1$,

$$\begin{aligned}
&\mathbb{E}_P(F_N \mid \mathcal{F}_{\tilde{v}_i}) = \mathbb{E}_P\Big(\mathbb{E}_P(F_N \mid \mathcal{F}_{\tilde{v}_i^*}) \mid \mathcal{F}_{\tilde{v}_i}\Big) \\
&\ge \mathbb{E}_P\Big(U_i^{\mathrm{m}}(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i, \omega(\tilde{v}_i^*)) \mid \mathcal{F}_{\tilde{v}_i}\Big) - (N-i)(3C+3)\epsilon + (C+1)\epsilon \\
&= \mathbb{E}\, U_i^{\mathrm{m}}(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i, \xi) - (N-i)(3C+3)\epsilon + (C+1)\epsilon \\
&\ge U_i^{\mathrm{e}}(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i) - (N-i)(3C+3)\epsilon \quad \text{a.s.}
\end{aligned}$$

where $\xi$ is the value at time $S/N - \Delta$ (rather than $S/N$ as in the definition of $U_i^{\mathrm{e}}$) of a linear Brownian motion started at $\tilde{x}_i$ at time 0 and stopped when it hits level 0, and $\mathbb{E}$ (without a subscript) refers to averaging over $\xi$ only. The last inequality can be derived as follows:

- Using the time period $[0, S/N - \Delta]$ in place of $[0, S/N]$ in the definition of $\xi$, we make an error (in the value of $\xi$) of at most $\delta$ with probability at least $1 - \epsilon$: cf. (31).

- This leads to an error of at most $f(\delta)$ with probability at least $1-\epsilon$ in the expression $\mathbb{E}\,U_i^{\mathrm{m}}(\tilde{x}_1,\tilde{v}_1,\ldots,\tilde{x}_i,\tilde{v}_i,\xi)$, where $f$ is a modulus of continuity for all $U_i^{\mathrm{m}}$, $i=0,\ldots,N-1$.
- Without loss of generality assume $f(\delta)\le\epsilon$.

#### 3.2.5 Tackling game-theoretic probability

Now we show that $\overline{\mathbb{E}}^{\mathrm{g}}(F_N)\le U_0^{\mathrm{e}}$.

Let $\epsilon>0$ be a small positive number (see below for details of how small), let $L$ be a large positive integer (see below for details of how large depending on $\epsilon$), and for each $i=N,N-1,\ldots,0$, define a function

$$\overline{U}_i:\mathbb{N}_0\times\{0,1,\ldots,L\}\times D_i^{\mathrm{e}}\to[0,\infty)$$

by

$$\overline{U}_i(X,L;x_1,v_1,\ldots,x_i,v_i):=U_i^{\mathrm{m}}(x_1,v_1,\ldots,x_i,v_i,X\sqrt{S/NL}) \qquad (34)$$

and, for $j=L-1,\ldots,1,0$,

$$\overline{U}_i(X,j;x_1,v_1,\ldots,x_i,v_i):=\\ \frac{\overline{U}_i(X-1,j+1;x_1,v_1,\ldots,x_i,v_i)+\overline{U}_i(X+1,j+1;x_1,v_1,\ldots,x_i,v_i)}{2}, \qquad (35)$$

if $X>0$, and

$$\overline{U}_i(0,j;x_1,v_1,\ldots,x_i,v_i):=\overline{U}_i(0,j+1;x_1,v_1,\ldots,x_i,v_i). \qquad (36)$$

Equations (34)–(36) assume $v_i<1$; if $v_i=1$, set, e.g.,

$$\overline{U}_i(X,j;x_1,v_1,\ldots,x_i,v_i):=U_i^{\mathrm{m}}(x_1,v_1,\ldots,x_i,v_i,X\sqrt{S/NL})$$

for all $j=0,\ldots,L$ (although the only interesting case for us is $v_i<1-\epsilon$). We will fix $i\in\{0,1,\ldots,N\}$ for a while.

Let us check that

$$U_i^{\mathrm{e}}(x_1,v_1,\ldots,x_i,v_i)\approx\overline{U}_i\left(\lfloor x_i/\sqrt{S/NL}\rfloor,0;x_1,v_1,\ldots,x_i,v_i\right), \qquad (37)$$

assuming $v_i<1$. This follows from the KMT theorem (Theorem 1 of Komlós, Major, and Tusnády [6]; see also [5]); we will use its following special case ([2], Theorem 1.5).

**KMT theorem.** *Let $E_1,E_2,\ldots$ be i.i.d. symmetric $\pm1$-valued random variables. For each $k$, let $S_k:=\sum_{i=1}^k E_i$. It is possible to construct a version of the sequence $(S_k)_{k\ge0}$ and a standard Brownian motion $(B_t)_{t\ge0}$ on the same probability space such that, for all $n$ and all $x\ge0$,*

$$\mathbb{P}\left(\max_{k\le n}|S_k-B_k|\ge C_1\ln n+x\right)\le C_2e^{-x},$$

*where $C_1$ and $C_2$ are absolute constants.*

(Although for our purpose much simpler results, such as those [9] based on Skorokhod's representation, would have been sufficient.) On the left-hand side of (37) we have the average of $\overline{U}_i^{\mathrm{m}}(x_1, v_1, \ldots, x_i, v_i, \cdot)$ w.r. to the value of a Brownian motion at time $S/N$ stopped when it hits level 0 and on the right-hand side of (37) we have the average of the same function w.r. to the value of a scaled simple random walk at the same time $S/N$ stopped when it hits level 0; the scaled random walk makes steps of $S/NL$ in time and $\sqrt{S/NL}$ in space; the Brownian motion and random walk are started from nearby points, namely $x_i$ and $\lfloor x_i/\sqrt{S/NL} \rfloor \sqrt{S/NL}$. By the KMT theorem there are coupled versions of the Brownian motion (not stopped) and the scaled simple random walk (also not stopped) that differ by at most $\epsilon$ over $[0, S]$ with probability at least $1 - \epsilon$, provided $L$ is large enough. (For example, we can take $L$ large enough for $x_i$ and $\lfloor x_i/\sqrt{S/NL} \rfloor \sqrt{S/NL}$ to be $\epsilon/2$-close and for the precision of the KMT approximation over $[0, S]$ to be $\epsilon/2$ with probability at least $1 - \epsilon$.) The values at time $S/N$ of the stopped Brownian motion and stopped scaled random walk can differ by more than $\epsilon$ even when their non-stopped counterparts differ by at most $\epsilon$ over $[0, S]$, but as the argument in Lemma 14 shows, the probability of this is at most $3\epsilon/\sqrt{S/N}$ (we would have $2\epsilon/\sqrt{S/N}$ if both coupled processes were Brownian motions, and replacing 2 by 3 adjusts for the discreteness of the random walk, for large $L$). Therefore, the difference between the two sides of (37) does not exceed

$$g(\epsilon) := f(\epsilon) + C3\epsilon/\sqrt{S/N}, \tag{38}$$

where $f$ is a modulus of continuity of $\overline{U}_i^{\mathrm{m}}$ for all $i = 0, \ldots, N-1$ and $C := \sup U$.

For $i = 1, \ldots, N$, set

$$v_i = v_i(\omega) := \phi_{iS/N}(\omega) \wedge 1, \tag{39}$$

$$x_i = x_i(\omega) := \omega(v_i). \tag{40}$$

During each non-empty time interval $[v_i(\omega), v_{i+1}(\omega))$ the trader will bet at the stopping times

$$T_{i,0}(\omega) := \inf\left\{t \geq v_i(\omega) \mid \omega(t)/\sqrt{S/NL} \in \mathbb{N}_0\right\},$$

$$T_{i,j}(\omega) := \inf\left\{t \geq T_{i,j-1}(\omega) \mid \omega(t)/\sqrt{S/NL} \in \mathbb{N}_0,\ \omega(t) \neq \omega(T_{i,j-1}(\omega))\right\}, \quad j \in \{1, \ldots, L\},$$

such that $T_{i,j}(\omega) < v_{i+1}(\omega) \wedge (1 - \epsilon)$; therefore, we are only interested in the case $j \in \{1, \ldots, J_i\}$ where

$$J_i = J_i(\omega) := \max\bigl\{j \in \{0, \ldots, L\} \mid T_{i,j}(\omega) < v_{i+1}(\omega)\bigr\}$$

($J_i = L$ being a common case). Besides, the bet at the times $v_i(\omega)$ will be set to zero unless $v_i(\omega) = T_{i,0}(\omega)$. The bets at the times $T_{i,L}(\omega)$ will also be set to zero unless $T_{i,L}(\omega) = T_{i+1,0}(\omega)$.

For $j = 0, \dots, L$, set

$$X_{i,j} := \omega(T_{i,j}) / \sqrt{S/NL} \in \mathbb{N}_0.$$

The bet at time $T_{i,j}(\omega) < 1 - \epsilon$ is 0 if $X_{i,j} = 0$ or $j = L$; otherwise, it is defined in such a way that the increase of the capital over $[T_{i,j}, T_{i,j+1}]$ is typically

$$\overline{U}_i(X_{i,j+1}, j+1; x_1, v_1, \dots, x_i, v_i) - \overline{U}_i(X_{i,j}, j; x_1, v_1, \dots, x_i, v_i) \tag{41}$$

(this assumes, e.g., $T_{i,j+1} \le v_{i+1}$); namely, the bet at time $T_{i,j}$ is formally defined as

$$\frac{\overline{U}_i(X_{i,j}+1, j+1; x_1, v_1, \dots, x_i, v_i) - \overline{U}_i(X_{i,j}, j; x_1, v_1, \dots, x_i, v_i)}{\sqrt{S/NL}}. \tag{42}$$

(When $X_{i,j+1} > X_{i,j}$, the increase is (41) by the definition of the bet, and when $X_{i,j+1} < X_{i,j}$, the increase is (41) by the definition of the bet and the definition (35).)

Let us check that this strategy achieves the final value greater than or close to $F_N(\omega)$ (with high lower game-theoretic probability) starting from $U_0^{\mathrm{e}}$. More generally, we will check that the capital $\mathcal{K}$ of this strategy (started with $U_0^{\mathrm{e}}$) at time $v_i(\omega)$, $i = 0, 1, \dots, N$, satisfies

$$\mathcal{K}_{v_i(\omega)} \gtrsim U_i^{\mathrm{e}}\left(x_1(\omega), v_1(\omega), \dots, x_i(\omega), v_i(\omega)\right)$$

with lower game-theoretic probability close to 1, in the notation of (39)–(40). More precisely, we will check that, for $i = 0, 1, \dots, N$ such that $v_i(\omega) < 1 - \epsilon$,

$$\mathcal{K}_{v_i(\omega)} \ge U_i^{\mathrm{e}}\left(x_1(\omega), v_1(\omega), \dots, x_i(\omega), v_i(\omega)\right) - iA \tag{43}$$

with lower game-theoretic probability at least $1 - 2i\epsilon$, where

$$A := 3f(\epsilon) + g(\epsilon)$$

and $g(\epsilon)$ is defined by (38).

We use induction in $i$. Suppose (43) holds; our goal is to prove (43) with $i+1$ in place of $i$. We have, for $v_{i+1} < 1 - \epsilon$:

$$\begin{aligned}
\mathcal{K}_{v_{i+1}} &\ge \mathcal{K}_{T_{i,J_i}} - f(\epsilon) && (44)\\
&= \mathcal{K}_{T_{i,0}} + \overline{U}_i(X_{i,J_i}, J_i; x_1, v_1, \dots, x_i, v_i) - \overline{U}_i(X_{i,0}, 0; x_1, v_1, \dots, x_i, v_i)\\
&\qquad - f(\epsilon)\\
&\ge \mathcal{K}_{v_i} + \overline{U}_i(X_{i,J_i}, J_i; x_1, v_1, \dots, x_i, v_i) - U_i^{\mathrm{e}}(x_1, v_1, \dots, x_i, v_i) && (45)\\
&\qquad - f(\epsilon) - g(\epsilon)\\
&\ge \overline{U}_i(X_{i,J_i}, J_i; x_1, v_1, \dots, x_i, v_i) - iA - f(\epsilon) - g(\epsilon) && (46)\\
&\ge \overline{U}_i(X_{i,J_i}, L; x_1, v_1, \dots, x_i, v_i) - iA - 2f(\epsilon) - g(\epsilon) && (47)\\
&= U_i^{\mathrm{m}}(x_1, v_1, \dots, x_i, v_i, X_{i,J_i}\sqrt{S/NL}) - iA - 2f(\epsilon) - g(\epsilon) && (48)\\
&\ge U_i^{\mathrm{m}}(x_1, v_1, \dots, x_i, v_i, x_{i+1}) - iA - 3f(\epsilon) - g(\epsilon) && (49)\\
&\ge U_i^{\mathrm{e}}(x_1, v_1, \dots, x_i, v_i, x_{i+1}, v_{i+1}) - iA - 3f(\epsilon) - g(\epsilon) && (50)
\end{aligned}$$

where:

- the inequality (44) holds for a large enough $L$ and follows from the form (42) of the bets (called off at time $T_{i,L}$) and the uniform continuity of $U_i^{\mathrm{m}}$ (which propagates to $\overline{U}_i$) with $f$ as modulus of continuity (for all $i$); the error term $f(\sqrt{S/NL})$ is replaced by the cruder $f(\epsilon)$;
- the inequality (45) follows from the approximate equality (37), whose accuracy is given by (38) (notice that the accuracy (38) is also applicable to (37) with $\lceil\cdots\rceil$ in place of $\lfloor\cdots\rfloor$); this inequality also relies on the equality $\mathcal{K}_{T_{i,0}} = \mathcal{K}_{v_i}$, which follows from our definition of the bets;
- the inequality (46) holds with lower game-theoretic probability at least $1 - 2i\epsilon$ by the inductive assumption;
- the inequality (47) holds with lower game-theoretic probability at least $1 - \epsilon$ for a large enough $L$, and follows from Theorem 3.1 of [11] and the uniform continuity of $U_i^{\mathrm{m}}$ with $f$ as modulus of continuity;
- the equality (48) holds by the definition (34);
- the inequality (49) also holds with lower game-theoretic probability at least $1 - \epsilon$ for a large enough $L$ and follows from Theorem 3.1 of [11] and the uniform continuity of $U_i^{\mathrm{m}}$ with $f$ as modulus of continuity.

We can see that the overall chain (44)–(50) holds with lower probability at least $1 - 2(i+1)\epsilon$.

So far we have considered the case $v_{i+1} < 1 - \epsilon$. Now suppose

$$1 - \epsilon \in (v_i(\omega), v_{i+1}(\omega)].$$

As soon as time $1 - \epsilon$ is reached, the strategy stops playing: we will show that with a lower game-theoretic probability arbitrarily close to 1 the goal has been achieved. Indeed, as we saw above,

$$\mathcal{K}_{v_i(\omega)} \gtrsim \overline{U}_i^{\mathrm{e}}(x_1, v_1, \ldots, x_i, v_i)$$

with high lower game-theoretic probability. Let us check that

$$\mathcal{K}_{1-\epsilon} \gtrsim F_N(\omega)$$

with high lower game-theoretic probability. This is true since $\mathcal{K}_{1-\epsilon}$ is, with high lower probability, greater than or close to the average of

$$\begin{aligned}\overline{U}_i^{\mathrm{m}}(x_1, v_1, \ldots, x_i, v_i, \xi) &\geq \overline{U}_{i+1}^{\mathrm{e}}(x_1, v_1, \ldots, x_i, v_i, \xi, 1)\\ &= \overline{U}_{i+1}^{\mathrm{e}}(x_1, v_1, \ldots, x_i, v_i, \omega(1), 1)\\ &= \overline{U}_N^{\mathrm{e}}(x_1, v_1, \ldots, x_i, v_i, \omega(1), 1, \ldots, \omega(1), 1)\\ &\geq F_N(\omega) - f(\epsilon)\end{aligned}$$

(cf. (21) and (24)) over the value $\xi$ at time $(i+1)S/N - \langle\omega\rangle_{1-\epsilon}$ of a Brownian motion started at $\omega(1-\epsilon)$ at time 0 and stopped when it hits level 0, where $\langle\omega\rangle$ is the quadratic variation of $\omega$ as defined in [11], Section 8.

To ensure that his capital is always positive, the trader stops playing as soon as his capital hits 0. Increasing his initial capital by a small amount we can make sure that this will never happen (for $L$ sufficiently large). Increasing his initial capital by another small amount we can make sure that he always superhedges $F_N$ and not just with high lower game-theoretic probability. Letting $L \to \infty$, we obtain $\overline{\mathbb{E}}^{\mathrm{g}}(F_N) \le U_0^{\mathrm{e}}$.

# 4 Conclusion

There is no doubt that this version of the paper makes various unnecessary assumptions. To relax or eliminate those assumptions is a natural direction of further research.


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