A game-theoretic ergodic theorem for imprecise Markov chains

Gert de Cooman

Ghent University, SYSTeMS

gert.decooman@UGent.be
http://users.UGent.be/~gdcooma
gertekoo.wordpress.com

GTP 2014
CIMAT, Guanajuato
13 November 2014

My boon companions

FILIP HERMANS

ENRIQUE MIRANDA

JASPER DE BOCK

Jean Ville and martingales

The original definition of a martingale

Définition 1. — *Soit* $X_1, X_2, \ldots, X_n, \ldots$ *une suite de variables aléatoires, telle que les probabilités*

$$\text{Pr.}\{X_1 < x_1,\ X_2 < x_2,\ \ldots,\ X_n < x_n\} \qquad (n = 1, 2, 3, \ldots)$$

soient bien définies et que les X_i *ne puissent prendre que des valeurs finies.*

Soit une suite de fonctions $s_0,\ s_1(x_1),\ s_2(x_1, x_2),\ \ldots$ *non négatives telles que*

$$(14)\quad \begin{cases} s_0 = 1, \\ \mathfrak{M}_{x_1, x_2, \ldots, x_{n-1}}\{s_n(x_1, x_2, \ldots, x_{n-1}, X_n)\} = s_{n-1}(x_1, x_2, \ldots, x_{n-1}), \end{cases}$$

où $\mathfrak{M}_X\{Y\}$ *représente d'une manière générale la valeur moyenne conditionnelle de la variable* Y *quand on connaît la position du point aléatoire* X, *au sens indiqué par M. P. Lévy.*

Dans ces conditions, nous dirons que la suite $\{s_n\}$ *définit une martingale ou un jeu équitable.*

Étude critique de la notion de collectif, 1939, p. 83

In a (perhaps) more modern notation

Ville's definition of a martingale

A martingale s is a sequence of real functions s_o, $s_1(X_1)$, $s_2(X_1, X_2)$, ... such that

1. $s_o = 1$;
2. $s_n(X_1, \ldots, X_n) \geq 0$ for all $n \in \mathbb{N}$;
3. $E(s_{n+1}(x_1, \ldots, x_n, X_{n+1}) | x_1, \ldots, x_n) = s_n(x_1, \ldots, x_n)$ for all $n \in \mathbb{N}_0$ and all $x_1, \ldots, x_n$.

It represents the outcome of a fair betting scheme, without borrowing (or bankruptcy).

Ville's theorem

The collection of all (locally defined!) martingales determines the probability P on the sample space Ω:

$$P(A) = \sup\{\lambda \in \mathbb{R} \colon s \text{ martingale and } \limsup_{n\to+\infty} \lambda s_n(X_1,\ldots,X_n) \leq \mathbb{I}_A\}$$
$$= \inf\{\lambda \in \mathbb{R} \colon s \text{ martingale and } \liminf_{n\to+\infty} \lambda s_n(X_1,\ldots,X_n) \geq \mathbb{I}_A\}$$

Ville's theorem

The collection of all (locally defined!) martingales determines the probability P on the sample space Ω:

$$
\begin{aligned}
P(A) &= \sup\{\lambda \in \mathbb{R} \colon s \text{ martingale and } \limsup_{n \to +\infty} \lambda s_n(X_1, \ldots, X_n) \leq \mathbb{I}_A\} \\
&= \inf\{\lambda \in \mathbb{R} \colon s \text{ martingale and } \liminf_{n \to +\infty} \lambda s_n(X_1, \ldots, X_n) \geq \mathbb{I}_A\}
\end{aligned}
$$

Turning things around

Ville's theorem suggests that we could take a convex set of martingales as a primitive notion, and probabilities and expectations as derived notions.

That we need an convex set of them, elucidates that martingales are examples of partial probability assessments.

Imprecise probabilities: dealing with partial probability assessments

Partial probability assessments

lower and/or upper bounds for

- the probabilities of a number of events,
- the expectations of a number of random variables

Partial probability assessments

lower and/or upper bounds for

- the probabilities of a number of events,
- the expectations of a number of random variables

Imprecise probability models

A partial assessment generally does not determine a probability measure uniquely, only a convex closed set of them.

Partial probability assessments

lower and/or upper bounds for

- the probabilities of a number of events,
- the expectations of a number of random variables

Imprecise probability models

A partial assessment generally does not determine a probability measure uniquely, only a convex closed set of them.

IP Theory

systematic way of dealing with, representing, and making conservative inferences based on partial probability assessments

Lower and upper expectations

Lower and upper expectations

A Subject is uncertain about the value that a variable X assumes in $\mathscr{X}$.

Gambles:

A gamble $f\colon \mathscr{X} \to \mathbb{R}$ is an uncertain reward whose value is $f(X)$.
$\mathscr{G}(\mathscr{X})$ denotes the set of all gambles on $\mathscr{X}$.

Lower and upper expectations

A Subject is uncertain about the value that a variable X assumes in $\mathscr{X}$.

Gambles:

A gamble $f\colon \mathscr{X} \to \mathbb{R}$ is an uncertain reward whose value is $f(X)$.
$\mathscr{G}(\mathscr{X})$ denotes the set of all gambles on $\mathscr{X}$.

Lower and upper expectations:

A lower expectation is a real functional that satisfies:

E1. $\underline{E}(f) \geq \inf f$ [bounds]

E2. $\underline{E}(f+g) \geq \underline{E}(f) + \underline{E}(g)$ [superadditivity]

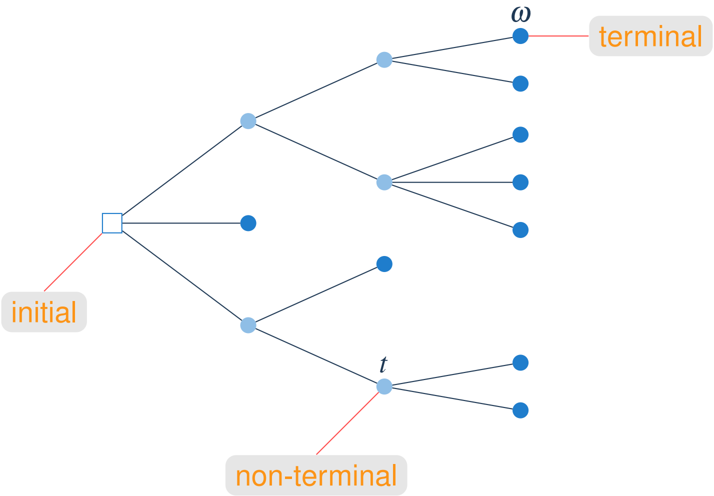
E3. $\underline{E}(\lambda f) = \lambda \underline{E}(f)$ for all real $\lambda \geq 0$ [non-negative homogeneity]

$\overline{E}(f) := -\underline{E}(-f)$ defines the conjugate upper expectation.

Sub- and supermartingales

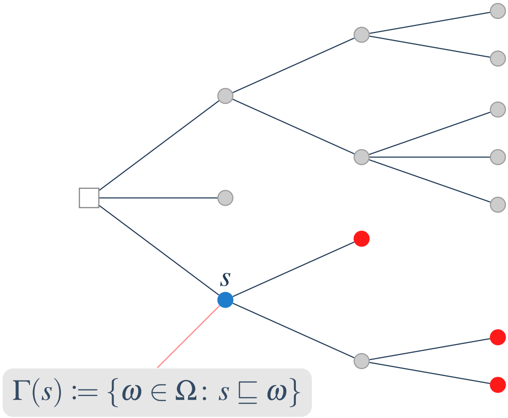
An event tree and its situations

Situations are nodes in the event tree, and the sample space Ω is the set of all terminal situations:



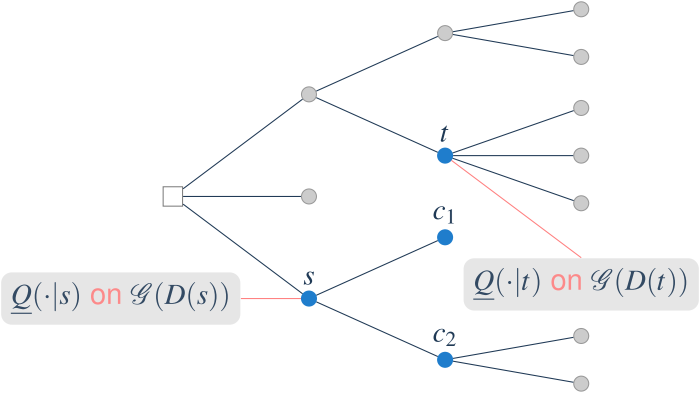
Events

An event A is a subset of the sample space Ω:



Local, or immediate prediction, models

In each non-terminal situation s, Subject has a belief model $\underline{Q}(\cdot|s)$.



$D(s) = \{c_1, c_2\}$ is the set of daughters of s.

Sub- and supermartingales

We can use the local models $\underline{Q}(\cdot|s)$ to define sub- and supermartingales:

A submartingale $\underline{\mathscr{M}}$

is a real process such that in all non-terminal situations s:

$$\underline{Q}(\underline{\mathscr{M}}(s\cdot)|s) \geq \underline{\mathscr{M}}(s).$$

A supermartingale $\overline{\mathscr{M}}$

is a real process such that in all non-terminal situations s:

$$\overline{Q}(\overline{\mathscr{M}}(s\cdot)|s) \leq \overline{\mathscr{M}}(s).$$

Lower and upper expectations

The most conservative lower and upper expectations on $\mathscr{G}(\Omega)$ that coincide with the local models and satisfy a number of additional continuity criteria (cut conglomerability and cut continuity):

Conditional lower expectations:

$$\underline{E}(f|s) := \sup\{\underline{\mathscr{M}}(s)\colon \limsup \underline{\mathscr{M}} \leq f \text{ on } \Gamma(s)\}$$

Conditional upper expectations:

$$\overline{E}(f|s) := \inf\{\overline{\mathscr{M}}(s)\colon \liminf \overline{\mathscr{M}} \geq f \text{ on } \Gamma(s)\}$$

Test supermartingales and strictly null events

A test supermartingale $\mathscr{T}$

is a non-negative supermartingale with $\mathscr{T}(\Box) = 1$.
(Very close to Ville's definition of a martingale.)

An event A is strictly null

if there is some test supermartingale $\mathscr{T}$ that converges to $+\infty$ on A:

$$\lim \mathscr{T}(\omega) = \lim_{n\to\infty} \mathscr{T}(\omega^n) = +\infty \text{ for all } \omega \in A.$$

If A is strictly null then

$$\overline{P}(A) = \overline{E}(\mathbb{I}_A) = \inf\{\overline{\mathscr{M}}(\Box)\colon \liminf \overline{\mathscr{M}} \geq \mathbb{I}_A\} = 0.$$

A few basic limit results

Supermartingale convergence theorem [Shafer and Vovk, 2001]

A supermartingale $\overline{\mathscr{M}}$ that is bounded below converges strictly almost surely to a real number:

$$\liminf \overline{\mathscr{M}}(\omega) = \limsup \overline{\mathscr{M}}(\omega) \in \mathbb{R} \text{ strictly almost surely.}$$

A few basic limit results

Strong law of large numbers for submartingale differences [De Cooman and De Bock, 2013]

Consider any submartingale $\underline{\mathscr{M}}$ such that its difference process

$$\Delta\underline{\mathscr{M}}(s) = \underline{\mathscr{M}}(s\,\cdot) - \underline{\mathscr{M}}(s) \in \mathscr{G}(D(s)) \text{ for all non-terminal } s$$

is uniformly bounded. Then $\liminf \langle\underline{\mathscr{M}}\rangle \geq 0$ strictly almost surely, where

$$\langle\underline{\mathscr{M}}\rangle(\omega^n) = \frac{1}{n}\underline{\mathscr{M}}(\omega^n) \text{ for all } \omega \in \Omega \text{ and } n \in \mathbb{N}$$

A few basic limit results

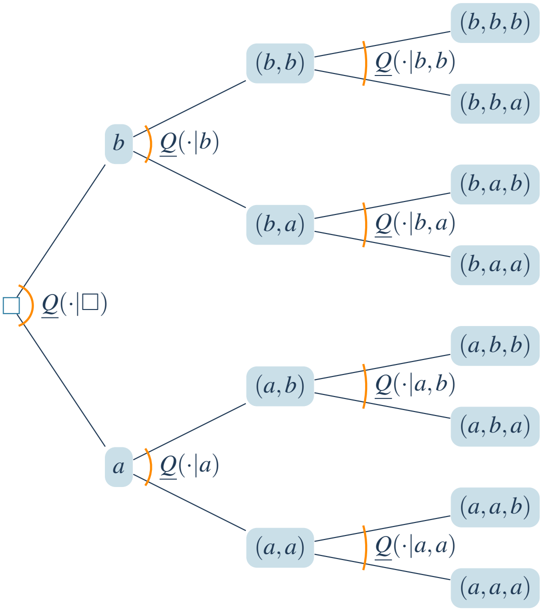
Lévy's zero–one law [Shafer, Vovk and Takemura, 2012]

For any bounded real gamble f on Ω:

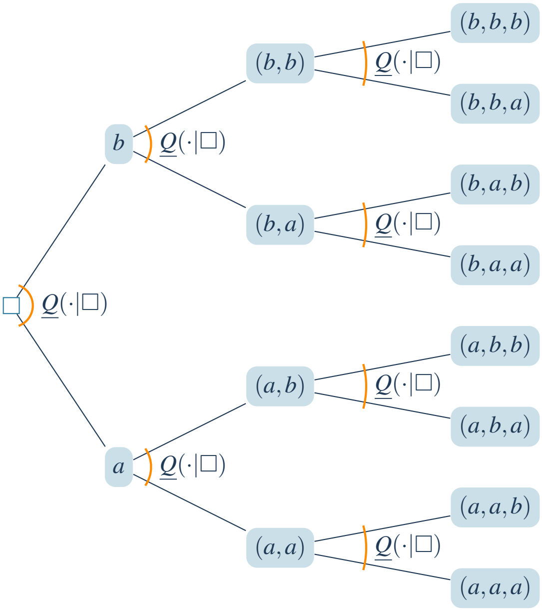
$$\limsup_{n\to+\infty} \underline{E}(f|\omega^n) \leq f(\omega) \leq \liminf_{n\to+\infty} \overline{E}(f|\omega^n) \text{ strictly almost surely.}$$

Imprecise Markov chains

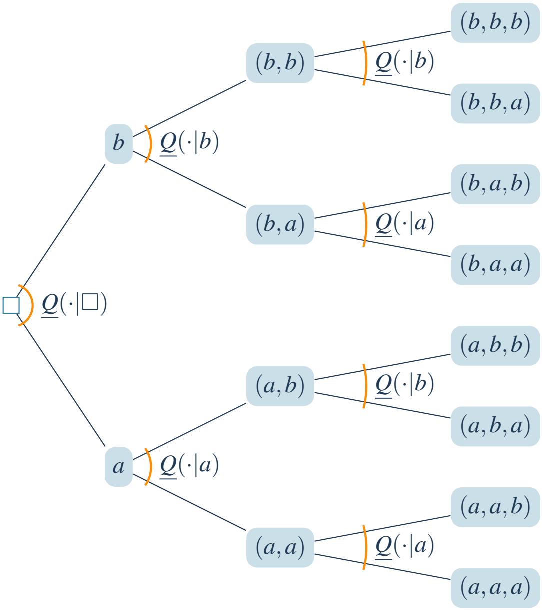
A simple discrete-time finite-state stochastic process



An imprecise IID model



An imprecise Markov chain



Stationarity and ergodicity

The lower expectation $\underline{E}_n$ for the state X_n at time n:

$$\underline{E}_n(f) = \underline{E}(f(X_n))$$

The imprecise Markov chain is Perron–Frobenius-like if for all marginal models $\underline{E}_1$ and all f:

$$\underline{E}_n(f) \to \underline{E}_\infty(f).$$

and if $\underline{E}_1 = \underline{E}_\infty$ then $\underline{E}_n = \underline{E}_\infty$, and the imprecise Markov chain is stationary.

In any Perron–Frobenius-like imprecise Markov chain:

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \underline{E}_n(f) = \underline{E}_\infty(f)$$

and

$$\underline{E}_\infty(f) \leq \liminf_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) \leq \limsup_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) \leq \overline{E}_\infty(f) \text{ str. almost surely.}$$

A more general ergodic theorem: the basics

Introduce a shift operator:

$$\theta\omega = \theta(x_1, x_2, x_3, \dots) := (x_2, x_3, x_4, \dots) \text{ for all } \omega \in \Omega,$$

and for any gamble f on Ω a shifted gamble $\theta f := f \circ \theta$:

$$(\theta f)(\omega) := f(\theta\omega) \text{ for all } \omega \in \Omega.$$

For any bounded gamble f on Ω, the bounded gambles:

$$g = \liminf_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \theta^k f \text{ and } g = \limsup_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \theta^k f$$

are shift-invariant: $\theta g = g$.

A more general ergodic theorem: use Lévy's zero–one law

In any Perron–Frobenius-like imprecise Markov chain, for any shift-invariant gamble $g = \theta g$ on Ω:

$$\lim_{n \to +\infty} \underline{E}(g|\omega^n) = \underline{E}_\infty(g) \text{ and } \lim_{n \to +\infty} \overline{E}(g|\omega^n) = \overline{E}_\infty(g)$$

and therefore

$$\underline{E}_\infty(g) \leq g \leq \overline{E}_\infty(g) \text{ strictly almost surely.}$$

New books

WILEY SERIES IN PROBABILITY AND STATISTICS

LOWER PREVISIONS

MATTHIAS C. M. TROFFAES | GERT DE COOMAN

WILEY

WILEY SERIES IN PROBABILITY AND STATISTICS

Introduction to Imprecise Probabilities

Edited by
Thomas Augustin   Frank P. A. Coolen
Gert de Cooman   Matthias C. M. Troffaes

WILEY