# A Closer Look at Adaptive Regret

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Royal Holloway, University of London

15/11/2014

# Outline

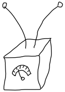
# Weather forecasting: adaptivity

Predictor 

Expert BBC

Expert Met Office

Expert itv

Nature 

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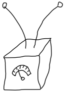
Nature 

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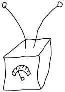
| Nature |  | |

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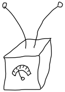
| Nature |  |  |

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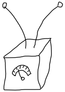
| Nature |  |  | |

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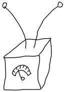
| Nature |  |  | |

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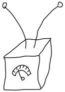
| Nature |  |  |  |

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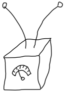
| Nature |  |  |  | |

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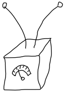
| Nature |  |  |  | |

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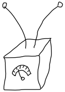
| Nature |  |  |  | ... |

Goal: close to the best expert overall (solution: AA)

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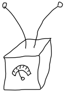
| Nature |  |  |  | ... | |

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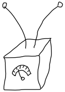
| Nature |  |  |  | ... | |

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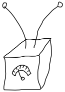
| Nature |  |  |  | ... | |

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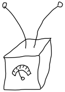
| Nature |  |  |  | ... | |

Goal: close to the best expert overall (solution: AA)

*Adaptive goal*: close to the best expert on every time interval

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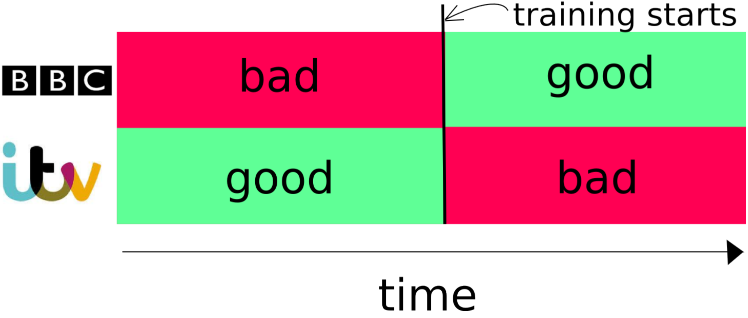
# Example continued



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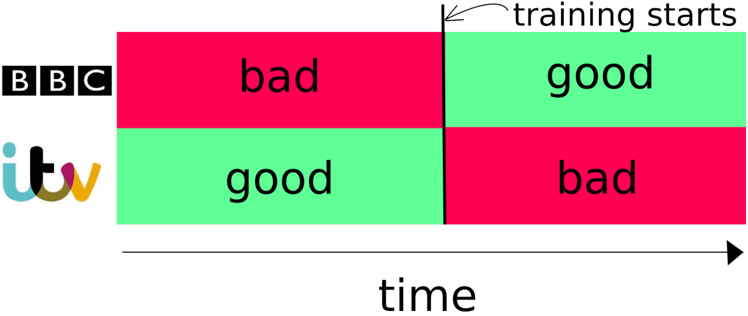


Non-adaptive predictor would lose trust in the first guy.

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We studied several approaches to adaptivity:

- Blowing up the set of experts to compete with virtual sleeping experts [DA, Koolen, Chernov, Vovk, 2012]

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## Our results

1. Figured out the Worst-Case adaptive regret of Fixed Share
2. Proved the optimality of Fixed Share — “no algorithm could have better guarantees on all time intervals”

# Protocol: Mix loss

**for** $t = 1, 2, \ldots$ **do**

Learner announces probability vector $\vec{w}_t \in \triangle_N$

Reality announces loss vector $\vec{\ell}_t \in [-\infty, \infty]^N$

Learner suffers loss $\ell_t := -\ln \sum_n w_t^n e^{-\ell_t^n}$

**end for**

# Adaptive Regret

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## Definition

The adaptive regret of the algorithm on the interval $[t_1, t_2]$ is the loss of the algorithm there minus the lost of the best expert there:

$$R_{[t_1,t_2]} := L_{[t_1,t_2]} - \min_j L^j_{[t_1,t_2]}$$

# AA and Fixed Share

Aggregating Algorithm [Vovk 1990] updates weights as:

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Fixed Share family is defined by the sequence of "switching rates" $\alpha_t$. Then the weight update is

$$w_{t+1}^n \; := \; \frac{\alpha_{t+1}}{N-1} + \left(1 - \frac{N}{N-1}\alpha_{t+1}\right) \frac{w_t^n e^{-\ell_t^n}}{\sum_n w_t^n e^{-\ell_t^n}} \,.$$

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Adaptivity hides in the first term.

# Fixed Share Wort-Case Adaptive regret data

We proved that the worst case data for Fixed Share looks like this:

| | 1 | … | $t_1 - 1$ | $t_1$ | … | $t_2$ |
|---|---|---|---|---|---|---|
| Expert 1 | ? | ? | ☠ | 0 | 0 | 0 |
| Expert 2 | ? | ? | ? | ☠ | ☠ | ☠ |
| … | ? | ? | ? | ☠ | ☠ | ☠ |
| Expert N | ? | ? | ? | ☠ | ☠ | ☠ |

where ☠ denotes infinite loss, 0 – zero loss and '?' – losses that don't matter.

# Fixed Share Worst-Case Adaptive regret formula

Knowing the worst-case data, we can plug it in and calculate the regret:

**Theorem**

*The worst-case adaptive regret of Fixed Share with $N$ experts on interval $[t_1, t_2]$ equals*

$$-\ln\left(\frac{\alpha_{t_1}}{N-1}\prod_{t=t_1+1}^{t_2}(1-\alpha_t)\right).$$

# Different $\alpha$-s: examples

- Classic Fixed Share ($\alpha_t = const$):

$$\begin{aligned} &\ln(N-1) - \ln\alpha - (t_2 - t_1)\ln\big(1-\alpha\big) && \text{for } t_1 > 1, \text{ and} \\ &\ln N - (t_2 - 1)\ln\big(1-\alpha\big) && \text{for } t_1 = 1. \end{aligned}$$

- Slowly decreasing $\alpha_t = 1/t$ leads to regret of

$$\begin{aligned} \ln(N-1) + \ln t_2 \qquad && \text{for } t_1 > 1, \text{ and} \\ \ln N + \ln t_2 \qquad && \text{for } t_1 = 1. \end{aligned}$$

# Different $\alpha$-s: examples

- Quickly decreasing switching rate.
  If we set $\alpha_t = t^{-2}$ we have the upper bound for regret

$$\ln N + 2 \ln t_1 + \ln 2.$$

  For $t_1 = 1$ this is very close to classical AA regret!

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| Expert 1 | 0 | 0 | ... | 0 | ☠ | ☠ | ... | ☠ | ☠ | ☠ | ... | ☠ |
| Expert 2 | ☠ | ☠ | ... | ☠ | 0 | 0 | ... | 0 | ☠ | ☠ | ... | ☠ |
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And the tracking bound can be recovered!

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**Theorem**

1. *(Any) Fixed Share is Pareto-optimal.*
2. *Any algorithm is dominated by an instance of Fixed Share.*

# Proof sketch – key lemma

- Let's call $\phi(t_1, t_2)$ a candidate guarantee. If $\phi(t_1, t_2)$ is witnessed by some algorithm as its worst-case regret we can prove the following bounds:

$$\begin{aligned}\phi(t,t) &\geq \ln N, \\ \phi(t_1,t_2) &\geq \phi(t_1,t_1) + \sum_{t=t_1+1}^{t_2} -\ln\left(1-(N-1)e^{-\phi(t,t)}\right)\end{aligned}$$

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\end{aligned}
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- Fixed Share with $\alpha_t = (N-1)\exp^{-\phi(t,t)}$ satisfies the last one with equality.

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# Summary

- We studied two intuitive methods to obtain adaptive algorithms.
- They turned out to be Fixed Share.
- The worst-case Adaptive Regret of Fixed Share was studied and its optimality was established.

Thank you!