Axiom E6 is needed in Proposition 6.9 of "Game-Theoretic Foundations for Probability and Finance"

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Abstract. In the winter of 2020-2021, when Covid-19 turned our lives around, Glenn Shafer gave a series of online lectures on "Game-theoretic foundations for statistical testing and imprecise probabilities" at the SIPTA school on imprecise probabilities. During those lectures, he mentioned a small open problem that intrigued him: in Proposition 6.9 of his new book with Vladimir Vovk, "Game-Theoretic Foundations for Probability and Finance", is Axiom $\mathrm{E6}^{[0,\infty]}$ needed or can it be removed? It intrigued me as well. In this short note, I provide a counterexample that demonstrates it cannot be removed.

Let $\mathcal{Y} := \{\mathrm{H}, \mathrm{T}\}$ and let $[0,\infty]^{\mathcal{Y}}$ be the set of all maps from $\mathcal{Y}$ to $[0,\infty]$—so all nonnegative extended real functions on $\mathcal{Y}$. Consider the operator $\overline{\mathrm{E}}$ on $[0,\infty]^{\mathcal{Y}}$ defined by

$$\overline{\mathrm{E}}(f) := \min\left\{\alpha \in [0,\infty] \colon f(\mathrm{H}) \leq \alpha, f(\mathrm{T}) \leq 2\alpha\right\} \text{ for all } f \in [0,\infty]^{\mathcal{Y}}.$$

This operator does not satisfy $\mathrm{E6}^{[0,\infty]}$ in [1, Proposition 6.9]. For example, if we let $\mathbf{1}_{\mathrm{T}} \in [0,\infty]^{\mathcal{Y}}$ denote the indicator of T, defined by $\mathbf{1}_{\mathrm{T}}(\mathrm{H}) := 0$ and $\mathbf{1}_{\mathrm{T}}(\mathrm{T}) := 1$, then $\overline{\mathrm{E}}(\mathbf{1}_{\mathrm{T}}) = 1/2$ and $\overline{\mathrm{E}}(\mathbf{1}_{\mathrm{T}} + 1) = 1$, hence $\overline{\mathrm{E}}(\mathbf{1}_{\mathrm{T}} + 1) \neq \overline{\mathrm{E}}(\mathbf{1}_{\mathrm{T}}) + 1$. So $\overline{\mathrm{E}}$ fails $\mathrm{E6}^{[0,\infty]}$ for $f = \mathbf{1}_{\mathrm{T}}$ and $c = 1$. It therefore follows from [1, Proposition 6.9] that $\overline{\mathrm{E}}$ is not a $[0,\infty]$-upper expectation on $\mathcal{Y}$.

On the other hand, it is fairly straightforward to show that $\overline{\mathrm{E}}$ satisfies Axioms $\mathrm{E1}^{[0,\infty]}$ to $\mathrm{E5}^{[0,\infty]}$ in [1, Proposition 6.9]. Hence, Axiom $\mathrm{E6}^{[0,\infty]}$ cannot be removed from [1, Proposition 6.9]. Interestingly though, as can be seen from [1, Proposition 6.8] and the definition of an upper expectation on p.113 of [1], this is not true if the role of $[0,\infty]$ is replaced by $\mathbb{R}$ or $\overline{\mathbb{R}}$. So the restriction to nonnegative functions seems to have a fundamental effect on the axioms that are needed to characterise (restrictions of) upper expectations.

References

1. Shafer, G., Vovk, V.: Game-Theoretic Foundations for Probability and Finance. Wiley, Hoboken, New Jersey (2019)