# Response to Freddy Delbaen's review of our book, *Probability and Finance: It's Only a Game*, in the September 2002 issue of the *Journal of the American Statistical Association* (Volume 97, number 459, p. 923)

**Glenn Shafer and Vladimir Vovk**

**January 6, 2003**

We are grateful that Professor Delbaen has reviewed our book. Seldom paid and often scorned, reviewers play an essential role in the dissemination and validation of new ideas. So we appreciate Professor Delbaen's willingness to grapple with a book that mingles heuristic and mathematical reasoning more than he likes. His review is friendly in many respects, and we take pleasure in its acknowledgement that we have something important and interesting to say.

The review also raises an interesting question about our book's game-theoretic framework for probability theory (how broad a range of behaviors by the market would permit arbitrage in our option pricing games), and it gives us an opportunity to comment on an aspect of the book that we would like to improve (our exposition of nonstandard analysis).

Unfortunately, the review discusses only a small fraction of the book's results, the ones closest to the measure-theoretic approach that the book challenges, and it fails to warn the reader about this narrow focus. This distresses us, because readers of the review may be left with the impression that the book is much less innovative and far ranging than it actually is. Readers can easily draw the following conclusions:

- The book is concerned only with results that hold with probability one.
- The first half of the book is only about coin tossing.
- The second half of the book considers only diffusion processes.

All of these conclusions are egregiously wrong. The review also contains several less serious but substantial factual errors.

Because it is so misleading, we cannot recommend Professor Delbaen's review to anyone who wants to know what our book is about. Fortunately, other reviews are available. A comprehensive and accurate description of the contents of the book, in a style suitable for professional mathematicians, can be found in Antonio Gualtierotti's review in *Mathematical Reviews* (2002k:60008). Other reviews and additional information, including some sample chapters, are available at http://www.cs.rhul.ac.uk/~vovk/book.

In the remainder of this document, we respond to a number of Professor Delbaen's comments one by one, in the order in which he makes them.

## Just a clever substitute for measure zero?

The first paragraph of Professor Delbaen's review reads as follows:

> This book is rather peculiar; it deals with probabilistic games and finance problems based on stochastic processes, but does so without probability. Of course, the authors mention a lot of probabilistic results and approaches, but they develop the theory based on games and strategies. Whether this can be done consistently is more a philosophical question than a mathematical one. In finance theory, many results are independent of the choice of a particular measure. Only the class of equivalent measures matters, and hence if there is a good substitute for the class of measure-zero sets, then there might be a way to present the finance results, and in fact many other game-theoretic or probabilistic results, by cleverly using this substitute. This is what the authors do in this book.

This passage seems good-natured, even sympathetic to the book it is reviewing, but it hugely misrepresents "what the authors do". It conveys the impression that most of the book is about measure-zero sets. In fact, most of the book has nothing at all to do with such sets.

Our book does include some asymptotic results, such as the strong laws in Part I and the Black-Scholes formula for diffusion processes in Part II, which say that certain events happen with game-theoretic probability one—i.e., except on a set of game-theoretic probability zero. In the simplest cases, coin tossing in Chapter 3 and diffusion with drift and volatility in Chapter 14 (and only in these simplest cases), the conclusion that these events happen with game-theoretic probability one truly is practically a rewording of the established conclusion that they happen with measure-theoretic probability one.

But most of the book—9 out of its 15 chapters—has little to nothing to do with sets of measure zero or with conclusions that hold with game-theoretic probability one:

- Four chapters (Chapters 10–13) are devoted exclusively to a new approach to hedging options, an approach that has no connection with sets of measure zero.

- Two chapters (Chapters 6 and 7) are concerned with the weak limit theorems, which tell us what happens with high probability, not what happens with probability one.

- Three chapters (Chapters 1, 2, and 9) are introductory and historical; they sometimes mention but do not emphasize results about measure zero.

Only four chapters are concerned with what happens with game-theoretic probability one: Chapters 3 and 4 on the strong law of large numbers, Chapter 5 on the iterated logarithm, and Chapter 14 on diffusion processes. Two other chapters, Chapters 8 and 15, are mixed, covering both strong and weak limit theorems. So the review's opening statement about "what the authors of this book do" is remotely relevant to at most one-third of the book—five out of fifteen chapters.

Even when they are applied only to the one-third of our book that is concerned with events with game-theoretic probability one, the suggestions made in the review's opening paragraph are seriously mistaken. In order to clarify this point, let us split these suggestions into two parts:

A. **Disguising Measure Zero.** It is suggested that we merely present existing results about sets of measure zero, cleverly changing their wording.

B. **Disguising Equivalent Probability Measures.** More specifically, it is suggested that when these results are put back in their usual form, they make assertions that are independent of the choice of an equivalent measure.

To assess the validity of these suggestions, we need to divide our results concerning game-theoretic probability one into four separate classes:

- The strong laws for coin tossing. (We devote three or four pages in Chapters 3 and 5 to this topic.) As we explain in Chapter 8 (pp. 179–180), coin tossing is the place where game-theoretic probability and measure-theoretic probability coincide. So Suggestion A is valid here. Suggestion B is not valid, however. Even for coin tossing, the strong limit theorems involve parameters like $\mathbb{E}(x_i \mid \mathcal{F}_{i-1})$, which do depend on the particular probability measure in the equivalence class.

- The strong laws more generally. (This is the topic of Chapters 3, 4, and 5.) Here not even Suggestion A is valid. As we explain in Chapter 8, our game-theoretic strong laws are stronger than the analogous measure-theoretic strong laws.

- Black-Scholes for a game-theoretic diffusion that prices drift and volatility. (We devote a couple of pages in Chapter 14 to this result.) Here both Suggestion A and Suggestion B are valid. As we say when we state the result on p. 345, "This is almost a direct translation of the measure-theoretic Black-Scholes formula."

- Black-Scholes for a game-theoretic diffusion that prices only volatility. (This is the main point of Chapter 14.) Here drift is not priced, and we say that the stock price is governed by the game-theoretic differential equation
$$dS(t) = \text{ no price for drift} + \sigma S(t) dW(t).$$
There is no measure-theoretic analogue for this model, and so neither Suggestion A nor Suggestion B makes sense here.

To summarize, we may say that Professor Delbaen's suggestions about how our results relate to existing results in measure-theoretic probability are valid only for a tiny portion of our results—a tiny portion that we explicitly labeled in our book as marking the overlap of our theory with the established theory.

## Event trees and filtrations

The fifth paragraph of the review reads:

> In the first half of the book, the model $\{-1,1\}^\infty$ is used, and there is a purely algebraic description (i.e., without reference to the Haar measure on this group) of the martingales. As a conclusion, it seems that measures are no longer needed. The games defined on the space $\{-1,1\}^\infty$ are defined in terms of event trees. I would have preferred to see the language of filtrations, which is logically equivalent but mathematically easier to handle. Also, the language of filtrations becomes inevitable in the second half of the book, so there is no reason to avoid it in the first part.

This paragraph is misleading on two rather broad fronts. It is true that $\{-1,1\}^\infty$ is used in Part I; it is the event tree for Reality's moves in a game in which Skeptic and Reality toss a fair coin. But the reader might be led to think that it is the event tree for Reality's moves in all the games in Part I, and this is far from true. Secondly, filtrations appear hardly at all in our exposition in Part II.

The reference to Haar measure (the uniform measure) on $\{-1,1\}^\infty$ can also mislead. Yes, $\{-1,1\}^\infty$ is a group, but this group structure is never used in the book. Coin tossing, represented by the event tree $\{-1,1\}^\infty$, is the core of classical probability theory, but the game-theoretic and measure-theoretic frameworks are two distinct generalizations of this classical core.

Finally, we do not want anyone to think that we are claiming filtrations can be replaced for all purposes by our event trees. Event trees do give rise to filtrations, but the mapping is neither one-to-one nor, in the case of continuous time, onto. These are two reasons why we do not use filtrations as our "language". Because the mapping is not one-to-one, a mere filtration does not provide all the structure our game-theoretic reasoning requires. Because it is not onto, an arbitrary filtration may fail to fit into our story at all. We do demonstrate, in Chapter 8, that the measure-theoretic versions of the classical limit theorems follow pretty directly from the game-theoretic versions, but this demonstration does not pretend that an arbitrary filtration can be represented straightforwardly by an event tree.

## Our nonstandard analysis

The next paragraph of the review begins:

> The second part of the book is less rigorous. There is extensive use of infinitesimal calculus, justified with nonstandard analysis. Whether nonstandard analysis is more intuitive than old-fashioned measure theory is a question that I do not want to discuss. Itô's lemma is used all of the time, but in the context of continuous processes that have quadratic variation. The stochastic calculus is in the same line as that of Föllmer (1981). ...

Part II of the book consists of seven chapters, numbered 9 through 15. Chapter 9 is a thoroughly heuristic introduction to the remaining chapters, and as we look back over it and reflect on Professor Delbaen's comments, we are forced to conclude that we should have signaled its heuristic nature more clearly at the outset. Chapters 10 and 15 do not use continuous time, and we believe that they are fully and transparently rigorous. The remaining four chapters, 11 through 14, are based, however, on nonstandard analysis.

The nonstandard analysis used in our book is not extensive; aside from the elementary properties of hyperreals, we use only a simple ultraproduct of games. This ultraproduct is explained in an appendix to Chapter 11, and we believe that readers who grasp the explanation will see that the arguments in Chapters 11 through 14 are in fact completely rigorous. But the reaction of Professor Delbaen and other readers with whom we have talked has persuaded us that our explanation of the ultraproduct is too brief. We plan to post a beefed-up explanation on the book's web site in the not too distant future. In the meantime, readers may gain some insight from lecture notes that Glenn Shafer has posted at www.glennshafer.com/lectures/lectures.html.

The assertion that we use Itô's lemma all the time is not accurate. This lemma comes into play only in a single chapter, Chapter 14. It is only in this chapter that we deal with a game that forces a price process to look something like a diffusion process, only here that our nonstandard analysis might be called a stochastic calculus. The games of Chapters 11 through 13 have much weaker rules, and it is here (and in the even more realistic but more complex discrete-time games of Chapter 10) that we see the greatest contribution of Part II: a framework for pricing options that does not assume that the price of the underlying asset behaves stochastically.

## Avoiding arbitrage

The paragraph just quoted continues:

> ... The approach of Shafer and Vovk works quite nicely when the processes are diffusion processes, but what it would do when the price process is continuous and has finite quadratic variation, but is not a semimartingale, is unclear. If the (say continuous) process is not a semimartingale, Delbaen and Schachermayer (1994, 1998) showed that there is a form of arbitrage. ...

This comment raises a very interesting question. It is obvious that if Market (a player in our games) generates the price (allowed to become negative) as a fractional Brownian motion with Hurst exponent in the range $1/2 < H \leq 1$, arbitrage is possible: since the price moves are positively correlated, a simple momentum strategy will do (such as the one in Shiryaev, 1999, Example 3 on p. 658). We can rephrase this by saying that in order to avoid arbitrage, Market must refrain from following fractional Brownian motion with Hurst exponent in this range. How much more can be said? What else must Market (who is not obliged to play stochastically) refrain from doing? How close dare he come to the forbidden fractional Brownian behavior?

## When does our Black-Scholes formula work?

The next paragraph begins:

> The authors illustrate their technique on the Black-Scholes formula and do so in quite some detail. First they do the heuristics, finding the pricing equation; then they prove that the option price so obtained is the game-theoretic price. ...

At this point we would like to explain to readers that more than technique is required to obtain our results in Chapters 10 through 13. We do not assume that the price of the underlying asset follows a stochastic process, but we do assume a radical change in the nature of options markets! We assume that rather than trading in calls and puts, these markets trade in a dividend-paying instrument—a derivative that pays a periodic (say daily) dividend equal to the observed volatility (the squared change or percentage change) of the price of the underlying during the period. We are proposing not only a technique but also a market reform.

## Martin's theorem

The penultimate paragraph of the review observes that we use the 1990 version of Martin's theorem and claims that we really need only the 1975 version. The claim is in error. Martin's 1990 result generalized his 1975 result from countable to uncountable trees. Our application of the theorem does involve uncountable trees, since it involves a game in which Reality chooses real numbers.

The reviewer also states that we only need Martin's theorem for Borel games. We do not need the greater generality of quasi-Borel games. This is correct. However, when Martin treated the uncountable case in 1990, he did not bother to state a theorem just for Borel games. (In the countable case, which he treated in 1975, he did talk about Borel games, but the distinction between Borel and quasi-Borel does not arise in this case.)

## Delbaen and Schachermayer

Professor Delbaen's final paragraph is concerned with Section 9.6 of our book, an appendix to our Chapter 9. The purpose of this appendix is explained in its introductory paragraph:

> In this appendix, we provide some additional information on stochastic option pricing and stochastic differential equations, aimed at readers new to these topics, who would like a fuller picture at the heuristic level of this chapter. First, we fill in some holes in our discussion of stochastic differential equations: we explain why (9.4)

represents a geometric Brownian motion, and we state Itô's lemma. Then we discuss what appears from the measure-theoretic point of view to be the general theory of option pricing: the theory of risk-neutral valuation.

The four and one-half pages that we devote to risk-neutral valuation are hardly adequate, of course, to say everything that Professor Delbaen would have liked us to say about this theory. He complains as follows:

> In dealing with the development of finance theory, and especially when dealing with no-arbitrage problems, the authors forgot to mention two basic papers, that of Kreps (1981) and the famous Harrison and Kreps paper of 1979. Also, the authors suggest that no-arbitrage is equivalent to the existence of risk-neutral (or martingale) measures—a statement that is not completely true, as the work of Delbaen and Schachermayer shows.

The final sentence is, of course, correct. In order to make the heuristic statement on p. 233 of our book ("The price process $\mathcal{S}(t)$ is arbitrage-free if and only if there exists at least one probability measure equivalent to $\mathbb{P}$ under which $\mathcal{S}$ is a martingale.") mathematically precise, we must add regularity conditions or at least replace "arbitrage-free" with the more complicated condition of "no free lunch with vanishing risk" formulated by Delbaen and Schachermayer (1994, 1998).

## References


Freddy Delbaen and Walter Schachermayer. 1994. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen* 300:463–520.

Freddy Delbaen and Walter Schachermayer. 1998. The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen* 312:215–250.

Hans Föllmer. 1981. Calcul d'Itô sans probabilités. In J. Azéma and M. Yor, editors, *Séminaire de Probabilités XV*, volume 850 of *Lecture Notes in Mathematics*, pp. 143–150. Springer, Berlin.

J. Michael Harrison and David M. Kreps. 1979. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory* 20:381–408.

David M. Kreps. 1981. Arbitrage and equilibrium in economies with infinitely many commodities. *Journal of Mathematical Economics* 8:15–35.

Albert N. Shiryaev. 1999. *Essentials of Stochastics in Finance: Facts, Models, Theory.* World Scientific, Singapore.