Pricing cones and pricing functionals

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Abstract

A pricing functional (also called a superexpectation functional) is determined by a cone of offers, and vice versa.

Let $\mathcal{Y}$ be a fixed set, which we will call the *outcome space*. We call a function $f : \mathcal{Y} \to (-\infty, \infty)$ a *payoff function*, and we write $(-\infty, \infty)^{\mathcal{Y}}$ for the set of all payoff functions.

We call a subset $\mathcal{C}$ of $(-\infty, \infty)^{\mathcal{Y}}$ a *pricing cone* if it satisfies the following conditions:

1. If $f \in \mathcal{C}$, $g \in (-\infty, \infty)^{\mathcal{Y}}$, and $g \leq f$, then $g \in \mathcal{C}$.
2. If $f \in \mathcal{C}$ and $c \in (0, \infty)$, then $cf \in \mathcal{C}$.
3. If $f, g \in \mathcal{C}$, then $f + g \in \mathcal{C}$.
4. If a sequence $f_1, f_2, \ldots$ of functions in $\mathcal{C}$ increases to a pointwise limit $f \in (-\infty, \infty)^{\mathcal{Y}}$, then $f \in \mathcal{C}$.
5. If $f \in \mathcal{C}$, then there exists $y \in \mathcal{Y}$ such that $f(y) \leq 0$.

We write $\mathbf{C}$ for the set of all pricing cones.

We call a function $\mathcal{E} : (-\infty, \infty)^{\mathcal{Y}} \to [-\infty, \infty]$ a *pricing functional* if it satisfies the following conditions:

1. If $f, g \in (-\infty, \infty)^{\mathcal{Y}}$ and $f \leq g$, then $\mathcal{E}(f) \leq \mathcal{E}(g)$.
2. If $f \in (-\infty, \infty)^{\mathcal{Y}}$ and $c \in (0, \infty)$, then $\mathcal{E}(cf) = c\mathcal{E}(f)$.
3. If $f, g \in (-\infty, \infty)^{\mathcal{Y}}$, then $\mathcal{E}(f + g) \leq \mathcal{E}(f) + \mathcal{E}(g)$ (with the right-hand side understood to be ∞ when $\mathcal{E}(f) = \infty$ or $\mathcal{E}(g) = \infty$).
4. If $c \in (-\infty, \infty)$, then $\mathcal{E}(c) = c$.
5. If a sequence $f_1, f_2, \ldots \in (-\infty, \infty)^{\mathcal{Y}}$ increases to a limit $f \in (-\infty, \infty)^{\mathcal{Y}}$, then
$$\mathcal{E}(f) = \lim_{k \to \infty} \mathcal{E}(f_k).$$

We write $\mathbf{E}$ for the set of all pricing functionals.

Lemma 1. *1. If $\mathcal{C}$ is a pricing cone, then the functional $\mathcal{E}^{\mathcal{C}} : (-\infty, \infty)^{\mathcal{Y}} \to [-\infty, \infty]$ defined by*

$$\mathcal{E}^{\mathcal{C}}(f) = \inf\{\alpha \,|\, (f - \alpha) \in \mathcal{C}\}$$

is a pricing functional.

2. *If $\mathcal{E}$ is a pricing functional, then the set $\mathcal{C}^{\mathcal{E}}$ defined by*

$$\mathcal{C}^{\mathcal{E}} = \{f \,|\, \mathcal{E}(f) \leq 0\}$$

is a pricing cone.

3. *If $\mathcal{E} \in \mathbf{E}$, then $\mathcal{E}^{(\mathcal{C}^{\mathcal{E}})} = \mathcal{E}$.*

4. *If $\mathcal{C} \in \mathbf{C}$, then $\mathcal{C}^{(\mathcal{E}^{\mathcal{C}})} = \mathcal{C}$.*

Here are some examples of pricing cones and the associated pricing functionals:

1. Suppose $\mathbb{P}$ is a probability measure on a σ-field of subsets of $\mathcal{Y}$, and write $\overline{\mathbb{E}}_{\mathbb{P}}(f)$ for the outer expected value of f with respect to $\mathbb{P}$. Then $\{f \in (-\infty, \infty)^{\mathcal{Y}} \mid \overline{\mathbb{E}}_{\mathbb{P}}(f) \leq 0\}$ is a pricing cone. The corresponding pricing functional is the functional $\overline{\mathbb{E}}_{\mathbb{P}}$.

2. Suppose $\mathcal{P}$ is a set of probability measures on $\mathcal{Y}$, possibly on different σ-algebras. Then $\{f \in (-\infty, \infty)^{\mathcal{Y}} \mid \overline{\mathbb{E}}_{\mathbb{P}}(f) \leq 0 \text{ for all } \mathbb{P} \in \mathcal{P}\}$ is a pricing cone. The corresponding pricing functional is given by $\mathcal{E}(f) = \sup_{\mathbb{P} \in \mathcal{P}} \overline{\mathbb{E}}_{\mathbb{P}}(f)$.

We call a function $f : \mathcal{Y} \to (-\infty, \infty]$ an *extended payoff function*, and we write $(-\infty, \infty]^{\mathcal{Y}}$ for the set of all extended payoff functions. We can extend a pricing functional to functions on $(-\infty, \infty]^{\mathcal{Y}}$ by taking limits; it will have the same properties.