Betting with negative probabilities

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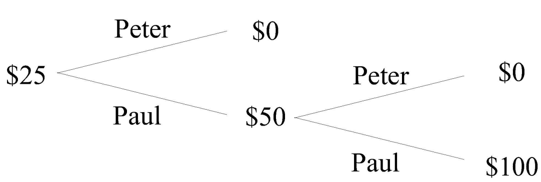

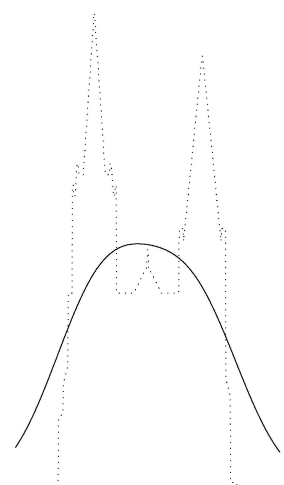

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Abstract

We demonstrate, on the example of Wigner's quasiprobability distribution, how negative probabilities can be treated and taken advantage of in the framework of game-theoretic probability.

Contents

1 Introduction

Negative probabilities may appear devoid of any empirical meaning. The frequentist semantics does not apply to them. “It is absurd to talk about an urn containing -17 red balls” [7, page 148]. Yet negative probabilities are usefully employed in quantum physics. The situation reminds us of the introduction of imaginary numbers in the theory of algebraic equations, even though the standard semantics, according to which numbers are quantities, does not apply to imaginary numbers.

In this note we illustrate that negative probabilities may be usefully employed in the game-theoretic approach to probabilities [9, 10].

2 Quantum measurement: statistical testing view

We work in the framework of the most standard formalization of (non-relativistic) quantum mechanics. It was originally proposed by John von Neumann [5, 6]. A modern, mathematically rigorous exposition is found in the book [4].

Like any physical system, a quantum system Q has a state space. The state space of Q is a Hilbert space $\mathcal{H}$. The states of Q are represented by unit vectors in $\mathcal{H}$.

For simplicity of exposition, we consider a quantum system Q of one particle Π moving in one dimension, though all of our results generalize to more (but finitely many) particles[1] moving in more (but finitely many) dimensions. The state space of our quantum system Q is the Hilbert space $L^2(\mathbb{R})$ of square integrable functions $f : \mathbb{R} \to \mathbb{C}$ with the inner product of $f, g \in L^2(\mathbb{R})$ given by the Lebesgue integral $\int f^*(t)g(t)dt$. Here $f^*(t)$ is the complex conjugate of $f^*(t)$, and, by default, the integrals are from $-\infty$ to $+\infty$.

Consider physical properties of Q, such as the position of particle Π, which take real values and can be measured. Such a physical property is represented by a self-adjoint operator A over $L^2(\mathbb{R})$. It is common to speak about measuring A itself and to call A an observable. The result of the measurement of (the physical property represented by) A in a given state ψ of Q is determined probabilistically. The probability that the result lies in a real interval $(u, v]$ is given by formula

$$\mathrm{Prob}_A(u, v] = \|(E_v - E_u)\psi\|^2 \tag{1}$$

where $\{E_r : r \in \mathbb{R}\}$ is the spectral resolution of the identity for A, whose existence follows from the spectral theorem for linear operators in Hilbert spaces proved originally by von Neumann [5]; a modern treatment is found in [4, §10].

We represent quantum measurement as the following perfect-information protocol of interaction (or game, except that we do not specify winning condi-

[1] The Wigner distribution (and the marginals that determine it) are about the instantaneous position and momentum of the (distinguishable, non-relativistic) particle(s). There is no time variable, no Hamiltonian. We don't have to worry about anything “happening”, such as a collision.

tions) between four players, Experimenter, Quantum Mechanics, Skeptic, and Reality. (The identities of Experimenter and Reality are not essential to us; we could combine them in one super player, World.)

Protocol 1 (Quantum measurement).

$\mathcal{K}_0 := 1$.

FOR $n = 1, 2, \dots,$

1. Experimenter prepares a state ψ_n of system Q and chooses an observable A_n to be measured in state ψ_n.
2. Quantum Mechanics provides a probability distribution μ_n on $\mathbb{R}$.
3. Skeptic chooses a measurable function $f_n \in [0, \infty]^{\mathbb{R}}$ such that $\int f_n d\mu_n = 1$.
4. Reality produces the result $r_n \in \mathbb{R}$ of the measurement of observable A_n in state ψ_n.
5. $\mathcal{K}_n := \mathcal{K}_{n-1} f_n(r_n)$.

Here Experimenter and Reality are free agents, who do not have to follow any strategy, deterministic or probabilistic. The strategy of Quantum Mechanics is given by formula (1), namely $\mu_n = \mathrm{Prob}_{A_n}$ at stage n. The goal of Skeptic is to test Quantum Mechanics, and we will be interested in strategies for testing available to Skeptic.

In Protocol 1, Skeptic tests Quantum Mechanics by gambling against its predictions. He starts from capital $\mathcal{K}_0 = 1$, and at time n his capital is $\mathcal{K}_n$. The condition $\int f_n d\mu_n = 1$ means that the game of testing is fair (from the point of view of Quantum Mechanics), and $f_n \geq 0$ means that Skeptic's capital is not allowed to become negative. Our interpretation is that $\mathcal{K}_n$ is the amount of evidence found by Skeptic against Quantum Mechanics (in general, against the null hypothesis). For further details of this style of testing, see [8] and [10, Chapter 1].

Protocol 1 is a simplified version of the protocols given in [10, §10.6] and [9, §8.4], which also involve the deterministic development of state ψ governed by the Schrödinger equation. Here is one corollary of game-theoretic limit theorems.

Corollary 1. *Let $F : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. Skeptic can force the event*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(F(r_n) - \int F d\mu_n \right) = 0,$$

in the sense of having a strategy ensuring $\mathcal{K}_n \to \infty$ whenever the equality fails.

Proof. This corollary, similarly to Corollary 10.14 in [10], can be deduced from Proposition 1.2 in [10]. Proposition 1.2 of [10] is a statement about Protocol 1.1 of [10], in which the restriction of Forecaster's and Reality's moves to $[-1,1]$ can be relaxed to requiring those moves to be bounded in absolute value by a given constant [10, p. 7]. A strategy for Skeptic whose existence is asserted in Corollary 1 can be obtained from any strategy Σ for Skeptic whose existence is asserted in Proposition 1.2 in [10] as follows:

- When playing in Protocol 1, feed Protocol 1.1 of [10] with Forecaster's moves $m_n := \int F d\mu_n$, Skeptic's moves recommended by Σ, and Reality's moves $y_n := F(r_n)$.
- When it is Skeptic's turn to make his move f_n in Protocol 1, he should set
$$f_n(r) := 1 + \frac{M_n}{\mathcal{K}_{n-1}} (F(r) - m_n) \qquad (2)$$
(and set, e.g., $f_n := 0$ if $\mathcal{K}_{n-1} = 0$), where m_n is defined as above, and M_n is the move recommended by Σ in response to the moves made so far in Protocol 1.1 of [10].

This will make sure that Skeptic's capital changes in the same way in Protocol 1 and in Protocol 1.1 of [10]: $\mathcal{K}_n := \mathcal{K}_{n-1} f_n(r_n)$ is equivalent to
$$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(y_n - m_n). \qquad \square$$

Corollary 1 (a law of large numbers) is unusual in that it lies outside Kolmogorov's framework for probability. The reason for that phenomenon is that Experimenter does not have to follow any strategy. Of course, real-world experimenters may follow deterministic or probabilistic testing strategies, which brings us into Kolmogorov's framework.

3 Wigner's quasiprobability distribution

Recall that our quantum system Q is a particle Π moving in one dimension. For technical reasons, we assume that Experimenter only prepares Q in states ψ which are smooth and compactly supported on $\mathbb{R}$; such states ψ will be called *nice*. Nice states are everywhere dense in $L^2(\mathbb{R})$. Below, by default, states of Q are nice.

Two important physical properties of Q are the position x and momentum p of particle Π. According to quantum mechanics, they are represented by self-adjoint operators
$$(X\psi)(x) := x\psi(x) \quad \text{and} \quad (P\psi)(x) := -i\hbar \frac{d\psi}{dx}(x)$$
respectively where $\hbar$ is a real constant, the so-called reduced Planck constant.

The uncertainty principle of quantum mechanics asserts a limit to the precision with which position x and momentum p can be determined simultaneously in a given state ψ, even if ψ is nice. You can know the distribution of x and that of p, but there is no joint probability distribution of x, p with the correct marginal distributions of x and p.

The situation changes if one allows negative probabilities. To address the issue, we need the following definitions. A *quasiprobability distribution* μ on a measurable space (Ω, Σ) is a real-valued, countably additive function on the measurable sets such that $\mu(\Omega) = 1$. A *quasiprobability density function* for a given quasiprobability distribution is the obvious generalization of a probability density function for a given probability distribution.

Remark 1. Quasiprobability distributions are special signed probability measures and are also known as *signed probability distributions*. One may worry whether countable additivity makes sense in signed probability spaces, but it does [3, §2.1].

In a 1932 paper [12], Eugene Wigner exhibited a function

$$W_\psi(x,p) := \frac{1}{2\pi} \int \psi^*\!\left(x + \frac{\beta\hbar}{2}\right) \psi\left(x - \frac{\beta\hbar}{2}\right) e^{i\beta p} d\beta$$

where ψ is an arbitrary unit vector in $L^2(\mathbb{R})$. It is easy to check that all values of Wigner's function $W_\psi(x,p)$ are real, but some values may be negative. In any nice state ψ, W_ψ gives rise to a unique quasiprobability distribution $\mathbf{W}_\psi$, *Wigner's quasiprobability distribution*, for which it is a quasiprobability density function.

Remark 2. The Wigner function W_ψ is also known as the Wigner-Ville function because it was introduced in 1948 by Jean-André Ville in the context of signal processing [2, 11]. Signal processing is beyond the scope of this paper.

For any real numbers a, b, the physical property $z = ax + bp$ of Q is represented by the self-adjoint operator $Z = aX + bP$. In any nice state ψ of the quantum system Q, let $w_\psi^{a,b}$ be the probability distribution Prob_Z given by formula (1) with $A = Z$.

The following proposition was presented in [1] and rigorously proved in [3].

Proposition 1. *In every nice state ψ, Wigner's quasiprobability distribution $\mathbf{W}_\psi$ is the unique quasiprobability distribution on $\mathbb{R}^2$ whose image, under any linear mapping $(x,p) \mapsto ax + bp$, is exactly $w_\psi^{a,b}$.*

Although $\mathbf{W}_\psi$ often has negative values, its images $w_\psi^{a,b}$ are genuine nonnegative probability distributions. The proposition allows us to refine Protocol 1 to the following perfect-information protocol.

Protocol 2 (Wigner-style quantum measurement)**.**

$\mathcal{K}_0 := 1$.

FOR $n = 1, 2, \ldots,$

1. Experimenter prepares a nice state ψ_n.
2. Quantum Mechanics provides a quasiprobability distribution, namely $\mathbf{W}_{\psi_n}$.
3. Experimenter chooses a pair (a_n, b_n) of real numbers and thus the observable $Z_n = a_n X + b_n P$.
4. Skeptic chooses a measurable function $f_n \in [0,\infty]^{\mathbb{R}}$ such that $\int f_n \, dw_{\psi_n}^{a_n,b_n} = 1$.
5. Reality produces the result $r_n \in \mathbb{R}$ of the measurement of observable Z_n in state ψ_n.
6. $\mathcal{K}_n := \mathcal{K}_{n-1} f_n(r_n)$.

Protocol 2 shows how we can test Wigner's quasiprobability distribution. Notice that Skeptic gambles only against nonnegative probability distributions $w_{\psi_n}^{a_n,b_n}$. This is a coherent testing protocol in the sense of [9, 10]. Recall that each probability distribution $w_{\psi_n}^{a_n,b_n}$ is an image of the quasiprobability distribution $\mathbf{W}_{\psi_n}$; negative probabilities are used only to generate nonnegative probabilities which are tested as usual.

Corollary 2. *Let $F : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. Skeptic can force the event*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(F(r_n) - \iint F(a_n x + b_n p) W_{\psi_n}(x,p) dx\, dp \right) = 0$$

in the sense of having a strategy ensuring $\mathcal{K}_n \to \infty$ whenever the equality fails.

Proof. Similarly to Corollary 1, this corollary will be also deduced from Proposition 1.2 of [10]. A strategy for Skeptic whose existence is asserted in Corollary 2 can be obtained from any strategy Σ for Skeptic whose existence is asserted in Proposition 1.2 in [10] as follows:

- When playing in Protocol 2, feed Protocol 1.1 of [10] with Forecaster's moves
$$m_n := \iint F(a_n x + b_n p) W_{\psi_n}(x,p) dx\, dp, \tag{3}$$
Skeptic's moves recommended by Σ, and Reality's moves $y_n := F(r_n)$.
- When it is Skeptic's turn to make his move f_n in Protocol 1, he should set (2), where m_n is now defined as (3), and M_n is still the move recommended by Σ in response to the moves made so far in Protocol 1.1 of [10].

The same argument as in the proof of Corollary 1 shows that Skeptic's capital changes in the same way in Protocol 1 and in Protocol 1.1 of [10].

It remains to show that f_n, as defined by (2), is a valid move for Skeptic, i.e., that $\int f_n \, dw_{\psi_n}^{a_n,b_n} = 1$. This follows from Proposition 1. $\square$

Corollary 2 is stronger than Corollary 1 in the sense that, in Protocol 2, Quantum Mechanics makes its move *before* Experimenter chooses an observable. By Proposition 1, this is impossible to achieve without negative probabilities.

4 Conclusion

Wigner's function is a simple and concise description of probabilistic predictions for a wide range of observables. It can be tested using the usual approach of game-theoretic probability. The function has found useful applications in physics [3] and signal processing [2], and we expect that the role of quasiprobability distributions will only grow both in practice and in the foundations of probability and statistics.

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