Rough paths in idealized financial markets (draft)

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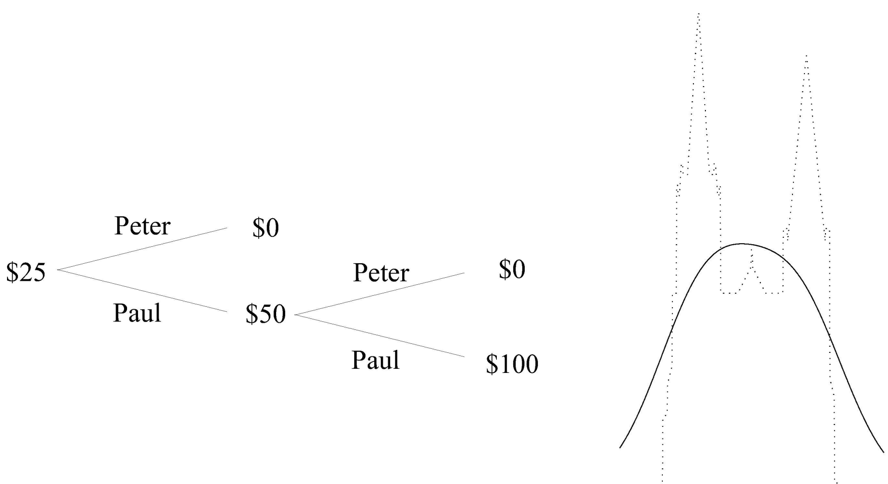

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Abstract

This note is a review of known results about the paths of security prices in idealized financial markets that satisfy a version of the no-arbitrage condition. Without making any probabilistic assumptions, it is sometimes possible to characterize the roughness of the price paths. A few simple new results are also stated.

Contents

1 Introduction

This note contains the references and proofs for my talks with the same name at the 3rd Workshop on Game-theoretic Probability and Related Topics (21 June 2010) and the 10th International Vilnius Conference on Probability and Mathematical Statistics, section "Random Processes", session "Rough Paths" (29 June 2010).

The study of rough paths "without probability" is an active field of research (Dudley and Norvaiša, Lyons,...). This note is a contribution to this field, studying price paths of financial securities in idealized markets. It comes from the tradition of "game-theoretic probability" (an approach to probability going back to von Mises and Ville). No probabilistic assumptions are made about the evolution of security prices (a non-stochastic notion of probability can be defined, but this step is optional). The early work on price paths in game-theoretic probability relied on using non-standard analysis (as in [15]); this note follows Takeuchi et al.'s recent paper [17] in avoiding non-standard analysis.

We will consider the price path of one financial security, in most of the note over a finite time interval $[0, T]$; all the results can be stated for $[0, \infty)$, but considering a finite time interval helps intuition. Our key assumption is that the market in our security is efficient, in the following weak sense (resembling the no-arbitrage condition): a prespecified trading strategy risking only 1 monetary unit (€1 for concreteness) will not bring infinite capital at time T. Our other assumption is that the interest rate over the time interval $[0, T]$ is 0, but it is easy to relax and made only for simplicity.

Let $\omega : [0, T] \to \mathbb{R}$ be the price path of our financial security. In Section 3 we consider the simplest case where ω is continuous; in this case there is no need to assume that $\omega \geq 0$ (although this assumption is usually satisfied in real markets). This is the main case, where our understanding is deepest. A typical result is that the p-variation of non-constant ω is finite when $p > 2$ and infinite when $p \leq 2$ (meaning that there is a trading strategy risking €1 that brings infinite capital at time T whenever this condition is violated). Section 2 discusses the case where ω is assumed to be càdlàg and positive (meaning $\omega \geq 0$); in this case it is only known that the p-variation of ω is finite when $p > 2$. In Section 4 we discuss the case where ω is only assumed right-continuous.

This note mainly reviews known results, but it also contains a few new ones. I will also state several open problems (which might well be easy for professionals).

Mathematical preliminaries and notation

The words "positive" and "increasing" are always understood in the wide sense of $\geq$; the adverb "strictly" will be added when needed.

For each $p \in (0, \infty)$, the *p-variation* $\mathrm{v}_p(f)$ of a function $f : [0, T] \to \mathbb{R}$ is defined as

$$\mathrm{v}_p(f) := \sup_{\kappa} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p , \qquad (1)$$

where n ranges over all strictly positive integers and κ over all *partitions* $0 = t_0 \le t_1 \le \cdots \le t_n = T$ of the interval $[0, T]$. It is obvious that, when f is bounded, there exists a unique number $\mathrm{vi}(f) \in [0, \infty]$, called the *variation index* of f, such that $\mathrm{v}_p(f)$ is finite when $p > \mathrm{vi}(f)$ and infinite when $p < \mathrm{vi}(f)$. It is easy to see that $\mathrm{vi}(f) \notin (0, 1)$ when f is continuous, but in general $\mathrm{vi}(f)$ can take any values in $[0, \infty]$.

2 Volatility of càdlàg price paths

Most of the work on "volatility without probability" has been done for continuous paths, but I will start my talk from the case of positive càdlàg price paths: since I do not have much to say about them, I will get this topic out of the way quickly.

Let Ω be the set of all positive càdlàg functions $\omega : [0, T] \to [0, \infty)$. For each $t \in [0, T]$, $\mathcal{F}_t^\circ$ is defined to be the smallest σ-algebra that makes all functions $\omega \mapsto \omega(s)$, $s \in [0, t]$, measurable; $\mathcal{F}_t$ is defined to be the universal completion of $\mathcal{F}_t^\circ$. A *process* S is a family of functions $S_t : \Omega \to [-\infty, \infty]$, $t \in [0, T]$, each S_t being $\mathcal{F}_t$-measurable (we drop the adjective "adapted"). An *event* is an element of the σ-algebra $\mathcal{F}_T$. Stopping times $\tau : \Omega \to [0, T] \cup \{\infty\}$ w.r. to the filtration $(\mathcal{F}_t)$ and the corresponding σ-algebras $\mathcal{F}_\tau$ are defined as usual; $\omega(\tau(\omega))$ and $S_{\tau(\omega)}(\omega)$ will be simplified to $\omega(\tau)$ and $S_\tau(\omega)$, respectively (occasionally, the argument ω will be omitted in other cases as well).

Remark. We define $\mathcal{F}_t$ to be the universal completion of $\mathcal{F}_t^\circ$ in order for the hitting times of closed sets in $\mathbb{R}$ to be stopping times [11], which will be used in the proof of Lemma 1 below.

Informal Remark. An alternative approach would be to define $\mathcal{F}_t := \mathcal{F}_{t+}^\circ$ (except that $\mathcal{F}_T := \mathcal{F}_T^\circ$) and to use the fact that the hitting times of open sets in $\mathbb{R}$ are stopping times. The disadvantage of this definition is that using the filtration $\mathcal{F}_{t+}^\circ$ allows "peeking ahead". It can be argued that in our context peeking ahead, just one instant into the future, is tolerable: since the price path is right-continuous, we can avoid peeking by updating our portfolio an instant later rather than now; the security price will not change. But perhaps not everybody will find this argument convincing, and so our definition does not use $\mathcal{F}_{t+}^\circ$.

The class of allowed trading strategies is defined in two steps. A *simple trading strategy* G consists of an increasing sequence of stopping times $\tau_1 \le \tau_2 \le \cdots$ and, for each $n = 1, 2, \ldots$, a bounded $\mathcal{F}_{\tau_n}$-measurable function h_n. It is required that, for any $\omega \in \Omega$, only finitely many of $\tau_n(\omega)$ should be finite. To such G and an *initial capital* $c \in \mathbb{R}$ corresponds the *simple capital process*

$$\mathcal{K}_t^{G,c}(\omega) := c + \sum_{n=1}^{\infty} h_n(\omega)\bigl(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)\bigr), \quad t \in [0, T] \tag{2}$$

(with the zero terms in the sum ignored); the value $h_n(\omega)$ will be called the *position* taken at time τ_n, and $\mathcal{K}_t^{G,c}(\omega)$ will sometimes be referred to as the capital process of G started with c.

A *positive capital process* is any process S that can be represented in the form

$$S_t(\omega) := \sum_{m=1}^{\infty} \mathcal{K}_t^{G_m,c_m}(\omega), \tag{3}$$

where the simple capital processes $\mathcal{K}_t^{G_m,c_m}(\omega)$ are required to be positive, for all t and ω, and the positive series $\sum_{m=1}^{\infty} c_m$ is required to converge. The sum (3) is always positive but allowed to take value ∞. Since $\mathcal{K}_0^{G_m,c_m}(\omega) = c_n$ does not depend on ω, $S_0(\omega)$ also does not depend on ω and will sometimes be abbreviated to S_0. In our discussions we will sometimes refer to the sequence $(G_m, c_m)_{m=1}^{\infty}$ as a *trading strategy* risking $\sum_m c_m$ and refer to (3) as the *capital process* of this strategy. (So that in this case the initial capital is regarded as part of the strategy.)

Informal Remark. The intuition behind the definition of positive capital processes is that the initial capital is split into infinitely many accounts and the trader runs a separate simple trading strategy on each of these accounts. Our definition of simple trading strategies only involves the position taken in security, not the cash position. The cash position is determined uniquely from the condition that the strategy should be self-financing (see p. 11 for further details) and in many questions there is no need to mention it explicitly.

We say that $E \subseteq \Omega$ is *null* if there is a positive capital process that starts from 1 and tends to ∞ on E. A property of $\omega \in \Omega$ will be said to hold *almost surely* (a.s.), or for *almost all* ω, if the set of ω where it fails is null. Intuitively, we expect such a property to be satisfied in a market that is efficient at least to some degree.

Theorem 1. *For almost all* $\omega \in \Omega$,

$$\mathrm{vi}(\omega) \le 2. \tag{4}$$

In the case of semimartingales, the property (4) was established by Lepingle ([10], Theorem 1(a)). Intuitively, Theorem 1 says that the price paths cannot be too rough. In fact, this theorem, and all other results of this kind in this note, can be strengthened to say that there is a trading strategy risking at most €1 whose capital process is ∞ at any time t such that the volatility index of ω over $[0, t]$ is greater than 2.

Theorem 1 will be proved using Stricker's [16] method (which is an extension of Bruneau's [2] method from continuous to càdlàg functions).

Let $\mathrm{M}_a^b(f)$ (resp. $\mathrm{D}_a^b(f)$) be the number of upcrossings (resp. downcrossings) of the open interval (a, b) by a function $f : [0, T] \to \mathbb{R}$ during the time interval $[0, T]$. For each $h > 0$ set

$$\mathrm{M}(f, h) := \sum_{k\in\mathbb{Z}} \mathrm{M}_{kh}^{(k+1)h}(f), \quad \mathrm{D}(f, h) := \sum_{k\in\mathbb{Z}} \mathrm{D}_{kh}^{(k+1)h}(f).$$

The key ingredient of the proof of Theorem 1 is the following game-theoretic version of Doob's upcrossings inequality:

Lemma 1. *Let $0 \le a < b$ be real numbers. There exists a positive simple capital process S that starts from $S_0 = a$ and satisfies, for all $\omega \in \Omega$,*

$$S_T(\omega) \ge (b-a)\,\mathrm{M}_a^b(\omega).$$

Proof. The following standard argument is easy to formalize. A simple trading strategy G leading to S (with initial capital a) can be defined as follows. At first G takes position 0. When ω first hits $[0,a]$, G takes position 1 until ω hits $[b,\infty)$, at which point G takes position 0; after ω hits $[0,a]$, G maintains position 1 until ω hits $[b,\infty)$, at which point G takes position 0; etc. Since ω is positive, S will also be positive. □

In fact, in Proposition 1 below we will prove a stronger version of Theorem 1. But to state the stronger version we will need a generalization of the definition (1). Let $\phi : [0,\infty) \to [0,\infty)$. For $f : [0,T] \to \mathbb{R}$, we set

$$\mathrm{v}_\phi(f) := \sup_\kappa \sum_{i=1}^n \phi\left(|f(t_i) - f(t_{i-1})|\right), \tag{5}$$

where κ ranges over all partitions $0 = t_0 \le t_1 \le \cdots \le t_n = T$ of $[0,T]$. Define $\ln^* u := 1 \vee |\ln u|$.

Proposition 1. *Set $\phi(u) := (u/\ln^* u)^2$. Then $\mathrm{v}_\phi(\omega) < \infty$ a.s.*

Our proof will show that Proposition 1 continues to hold if we define $\phi(u) := u^2/(\ln^* u \ln^* \ln^* u \cdots)$. The inequality $\mathrm{v}_\phi(\omega) < \infty$ a.s., however, is still much weaker than the inequality $\mathrm{v}_\psi(\omega) < \infty$ a.s., with ψ defined by (15), that we can prove assuming ω continuous (see Corollary 1 below).

Proof of Proposition 1. Choose a decreasing function $w : \mathbb{R} \to (0,\infty)$ such that $\sum_{j=0}^\infty w(j) = 1$ (such as $w(u) \propto (u \vee 1)^{-2}$).

Let $0 = t_0 \le t_1 \le \cdots \le t_n = T$ be a partition of the interval $[0,T]$; without loss of generality we replace all "$\le$" by "$<$". Fix $\omega \in \Omega$ and $\lambda \ge \sup_{t\in[0,T]} \omega(t)$. Split

$$\sum_{i=1}^n \phi\left(|\omega(t_i) - \omega(t_{i-1})|\right)$$

into two parts:

$$\sum_{i=1}^n \phi\left(|\omega(t_i) - \omega(t_{i-1})|\right) = \sum_{i\in I_+} \phi\left(\omega(t_i) - \omega(t_{i-1})\right) + \sum_{i\in I_-} \phi\left(\omega(t_{i-1}) - \omega(t_i)\right),$$

where

$$I_+ := \{i \mid \omega(t_i) - \omega(t_{i-1}) > 0\},$$

$$I_- := \{i \mid \omega(t_i) - \omega(t_{i-1}) < 0\}.$$

By Lemma 1, for each $j = 0, 1, \dots$ and each $k \in \{0, \dots, 2^j - 1\}$ there exists a positive simple capital process $S^{j,k}$ that starts from $k\lambda 2^{-j}$ and satisfies

$$S_T^{j,k}(\omega) \ge \lambda 2^{-j} \, \mathrm{M}_{k\lambda 2^{-j}}^{(k+1)\lambda 2^{-j}}(\omega).$$

Summing $2^{-j} S^{j,k}$ over $k = 0, \dots, 2^j - 1$, we obtain a positive simple capital process S^j such that

$$S_0^j = \sum_{k=0}^{2^j - 1} k\lambda 2^{-2j} \le \lambda/2,$$
$$S_T^j(\omega) \ge \lambda 2^{-2j} \, \mathrm{M}(\omega, \lambda 2^{-j}).$$

For each $i \in I_+$, let $j(i)$ be the smallest (positive) integer j satisfying

$$\exists k \in \{0, 1, \dots\} : \omega(t_{i-1}) \le k\lambda 2^{-j} \le (k+1)\lambda 2^{-j} \le \omega(t_i).$$

Summing $w(j) S^j$ over $j = 0, 1, \dots$, we obtain a positive capital process S such that

$$\begin{aligned}
S_0 &\le \lambda/2, \\
S_T(\omega) &\ge \frac{1}{\lambda} \sum_{j=0}^{\infty} w(j)(\lambda 2^{-j})^2 \, \mathrm{M}(\omega, \lambda 2^{-j}) \ge \frac{1}{\lambda} \sum_{i \in I_+} w(j(i))(\lambda 2^{-j(i)})^2 \qquad (6) \\
&\ge \frac{1}{4\lambda} \sum_{i \in I_+} w\left(-\log\left(\frac{\omega(t_i) - \omega(t_{i-1})}{4\lambda}\right)\right) (\omega(t_i) - \omega(t_{i-1}))^2 \\
&\ge \delta \sum_{i \in I_+} \phi\left(\omega(t_i) - \omega(t_{i-1})\right), \qquad (7)
\end{aligned}$$

where log is binary logarithm, $\delta > 0$ depends only on λ, and

$$\phi(u) := u^2 w(-\log u).$$

(The second inequality in (6) follows from the fact that to each $i \in I_+$ corresponds an upcrossing of an interval of the form $[k\lambda 2^{-j(i)}, (k+1)\lambda 2^{-j(i)}]$.)

An inequality analogous to the inequality between the second and the last terms of the chain (6)–(7) can be proved for downcrossings instead of upcrossings, I_- instead of I_+, and $\omega(t_{i-1}) - \omega(t_i)$ instead of $\omega(t_i) - \omega(t_{i-1})$. Using this inequality (in the third "$\ge$" below) gives

$$\begin{aligned}
S_T(\omega) &\ge \frac{1}{\lambda} \sum_{j=0}^{\infty} w(j)(\lambda 2^{-j})^2 \, \mathrm{M}(\omega, \lambda 2^{-j}) \\
&\ge \frac{1}{\lambda} \sum_{j=0}^{\infty} w(j)(\lambda 2^{-j})^2 \left(\mathrm{D}(\omega, \lambda 2^{-j}) - 1\right)
\end{aligned}$$

$$\geq \delta \sum_{i \in I_-} \phi\left(\omega(t_{i-1}) - \omega(t_i)\right) - \lambda \sum_{j=0}^{\infty} w(j) 2^{-2j}.$$

Averaging the two lower bounds for $S_T(\omega)$, we obtain

$$S_T(\omega) \geq \frac{\delta}{2} \sum_{i=1}^{n} \phi\left(|\omega(t_i) - \omega(t_{i-1})|\right) - \frac{\lambda}{2} \sum_{j=0}^{\infty} w(j) 2^{-2j}.$$

Taking the supremum over all partitions gives

$$S_T(\omega) \geq \frac{\delta}{2} \operatorname{v}_\phi(\omega) - \frac{\lambda}{2} \sum_{j=0}^{\infty} w(j) 2^{-2j}.$$

We can see that the event that $\sup \omega \leq \lambda$ and $\operatorname{v}_\phi(\omega) = \infty$ is null. Therefore, the event that $\operatorname{v}_\phi(\omega) = \infty$ is also null. □

The case of positive càdlàg price paths considered in this section is very different from the case of continuous price paths that we take up in the following section: Theorem 2 will show that, in the latter case, $\operatorname{vi}(\omega) \in \{0, 2\}$ a.s. In the former case, all price paths with finite variation are allowed, as the following proposition will show.

The *upper probability* of a set $E \subseteq \Omega$ is defined as

$$\overline{\mathbb{P}}(E) := \inf\left\{S_0 \mid \forall \omega \in \Omega : S_T(\omega) \geq \mathbb{I}_E(\omega)\right\}, \tag{8}$$

where S ranges over the positive capital processes and $\mathbb{I}_E$ stands for the indicator of E. In this section we will be interested only in one-element sets E. We write $\operatorname{v}(f)$ meaning $\operatorname{v}_1(f)$.

Proposition 2. *For any $\omega \in \Omega$,*

$$\overline{\mathbb{P}}(\{\omega\}) = \sqrt{\frac{\omega(0)}{\omega(T)} e^{-\operatorname{v}(\ln \omega)}}. \tag{9}$$

Proof. Let S be any positive capital process. Represent it in the form (3). It suffices to prove that none of the component strategies G_m can increase the initial capital c_m by more than a factor of

$$\sqrt{\frac{\omega(T)}{\omega(0)} e^{\operatorname{v}(\ln \omega)}}$$

and that this factor itself is attainable, at least in the limit. Fix an m and let $\tau_1, \tau_2, \ldots$ and $h_1, h_2, \ldots$ be the component stopping times and positions of G_m. It is clear that all h_n must be positive in order for $\mathcal{K} := \mathcal{K}^{G_m, c_m}$ to be positive: upward price movements are unbounded. Downward price movements after τ_n can be as large as $\omega(\tau_n)$, which implies that

$$0 \leq h_n \leq \mathcal{K}_{\tau_n} / \omega(\tau_n).$$

This gives, according to (2),

$$\mathcal{K}_{\tau_{n+1}} = \mathcal{K}_{\tau_n} + h_n \left(\omega(\tau_{n+1}) - \omega(\tau_n)\right) \le \left(1 \vee \frac{\omega(\tau_{n+1})}{\omega(\tau_n)}\right) \mathcal{K}_{\tau_n}.$$

An optimal choice is

$$h_n := \begin{cases} \mathcal{K}_{\tau_n}/\omega(\tau_n) & \text{if } \omega(\tau_{n+1}) > \omega(\tau_n) \\ 0 & \text{otherwise.} \end{cases}$$

The optimal positive simple capital process increases its initial capital $e^{\mathrm{v}^+(\ln \omega)}$-fold, where $\mathrm{v}(f)$ is defined by the following modification of (1):

$$\mathrm{v}^+(f) := \sup_\kappa \sum_{i=1}^n \left(f(t_i) - f(t_{i-1})\right)^+ .$$

We can see that

$$\overline{\mathbb{P}}(\{\omega\}) = e^{-\mathrm{v}^+(\ln \omega)}. \tag{10}$$

If we define

$$\mathrm{v}^-(f) := \sup_\kappa \sum_{i=1}^n \left(f(t_i) - f(t_{i-1})\right)^- ,$$

we can see that $\mathrm{v}(f) = \mathrm{v}^+(f) + \mathrm{v}^-(f)$ and $f(T) - f(0) = \mathrm{v}^+(f) - \mathrm{v}^-(f)$; the last two equalities imply $\mathrm{v}^+(f) = (\mathrm{v}(f) + f(T) - f(0))/2$. In combination with (10), this gives (9). □

3 Volatility of continuous price paths

Let Ω be the set of all continuous functions $\omega : [0,T] \to \mathbb{R}$. Intuitively, this is the set of all possible price paths; now we do not insist that the price path should be positive. The σ-algebras $\mathcal{F}_t^\circ$ are defined as before, but now we simply set $\mathcal{F}_t := \mathcal{F}_t^\circ$ (there is no need for universal completion). The definitions of events, processes, positive capital processes, and null events are the same as before.

The following elaboration of Theorem 1 for continuous price paths was established in [19] using direct arguments (relying on the result in [2] mentioned earlier for the inequality $\mathrm{vi}(\omega) \le 2$ and a standard argument for the inequality $\mathrm{vi}(\omega) \ge 2$ for non-constant ω).

Theorem 2. *For almost all $\omega \in \Omega$,*

$$\mathrm{vi}(\omega) = 2 \text{ or } \omega \text{ is constant.} \tag{11}$$

This theorem is similar to the well-known property of continuous semimartingales (Lepingle [10], Theorem 1(a) and Proposition 3(b)). A probability-free result related to the inequality $\mathrm{vi}(\omega) \ge 2$ (for almost all non-constant ω) was established by Salopek [14] (p. 228), who proved that the trader can start from 0 and end up with a strictly positive capital in a market with two securities whose price paths ω_1 and ω_2 are strictly positive and satisfy $\mathrm{vi}(\omega_1) < 2$, $\mathrm{vi}(\omega_2) < 2$, $\omega_1(0) = \omega_2(0) = 1$ and $\omega_1(T) \neq \omega_2(T)$. However, Salopek's definition of the capital process only works under the assumption that all securities in the market have price paths ω satisfying $\mathrm{vi}(\omega) < 2$.

Intuitively, Theorem 2 seems to suggest that volatility is created by the process of trading itself, and not, for example, by news.

Open problem. Can anything similar to (11) be said in the case of positive càdlàg paths, as in the previous subsection? (From Proposition 2 we can see that (11) itself cannot be asserted a.s. in this case; see also Lemma 4 below.)

Theorem 2 says that, almost surely,

$$\mathrm{v}_p(\omega) \begin{cases} < \infty & \text{if } p > 2 \\ = \infty & \text{if } p < 2 \text{ and } \omega \text{ is not constant.} \end{cases}$$

The situation for $p = 2$ is clarified by a result from [20] which we will state below as Theorem 3; this result will also give an analogue of Taylor's [18] much more precise result. In particular, we will see that, for almost all non-constant ω, $\mathrm{v}_2(\omega) = \infty$, similarly to the case of continuous martingales (see [10], proof of Proposition 3(b)).

Let us replace in the definitions given above the time interval $[0, T]$ by $[0, \infty)$ (this will make Theorem 3 easier to state); the only substantial changes are: now in the definition of simple trading strategies we require, for all ω, that $\tau_n(\omega) \to \infty$, instead of requiring $\tau_n(\omega) = \infty$ from some n on; the definition of upper probability is modified as in (13) below. At the beginning of the next subsection we will revert to the finite time interval $[0, T]$.

A *time change* is defined to be a continuous increasing (not necessarily strictly increasing) function $f : [0, \infty) \to [0, \infty)$ satisfying $f(0) = 0$. Equipped with the binary operation of composition, $(f \circ g)(t) := f(g(t))$, $t \in [0, \infty)$, the time changes form a (non-commutative) monoid, with the identity time change $t \mapsto t$ as the unit. The group of *space shifts* is the additive group of real numbers; we will consider it as a monoid. The monoid of *space/time changes* is the direct sum of the monoid of space shifts and the monoid of time changes. Each space/time change is a pair (c, f), where $c \in \mathbb{R}$ and f is a time change, and the product of space/time changes (c_1, f_1) and (x_2, f_2) is the space/time change $(c_1 + c_2, f_1 \circ f_2)$. The *action* of a space/time change (c, f) on $\omega \in \Omega$ is defined to be $\omega^{(c,f)} := c + \omega \circ f \in \Omega$. The *trail* of $\omega \in \Omega$ is the set of all $\phi \in \Omega$ such that $\phi^{(c,f)} = \omega$ for some space/time change (c, f). (In the standard case of monoids that are groups, trails are called orbits.)

A subset E of Ω is *space/time-superinvariant* if together with any $\omega \in \Omega$ it contains the whole trail of ω; in other words, if for each $\omega \in \Omega$ and each space/time change (c, f) it is true that

$$\omega^{(c,f)} \in E \Longrightarrow \omega \in E. \tag{12}$$

The *space/time-superinvariant class* $\mathcal{K}$ is defined to be the family of those events (elements of $\mathcal{F}_\infty$) that are space/time-superinvariant.

Remark. The space/time-superinvariant class $\mathcal{K}$ is a monotone class; however, simple examples show that it is not a σ-algebra.

The *upper probability* of a set $E \subseteq \Omega$ is now defined as

$$\overline{\mathbb{P}}(E) := \inf\bigl\{S_0 \bigm| \forall\omega \in \Omega : \liminf_{t\to\infty} S_t(\omega) \ge \mathbb{I}_E(\omega)\bigr\}, \tag{13}$$

where S ranges over the positive capital processes. It does not matter whether we write $\liminf_{t\to\infty}$, $\limsup_{t\to\infty}$, or $\sup_t$ in (13). Simple examples show that $\overline{\mathbb{P}}$ is not a probability measure, even if restricted to $\mathcal{F}_\infty$.

It is natural to say that $E \subseteq \Omega$ is null if $\overline{\mathbb{P}}(E) = 0$. This is equivalent to our "official" definition:

Lemma 2. *If $\overline{\mathbb{P}}(E) = 0$, there is a positive capital process that starts from 1 and tends to ∞ on E.*

Proof. Suppose $\overline{\mathbb{P}}(E) = 0$. For each $i = 1, 2, \dots$ there is a positive capital process that starts from 2^{-i} and whose lim inf is at least 1 on E. Sum these positive capital processes. □

Let $\mathcal{W}$ be the Wiener measure. The following theorem can be regarded as an analogue of the well-known Dubins–Schwarz result [8].

Theorem 3. *Each event $E \in \mathcal{K}$ satisfies*

$$\overline{\mathbb{P}}(E) \le \mathcal{W}(E). \tag{14}$$

Let $\psi : [0,\infty) \to [0,\infty)$ be Taylor's [18] function

$$\psi(u) := \frac{u^2}{2\ln^*\ln^* u}. \tag{15}$$

For $f : [0,\infty) \to \mathbb{R}$ and $T \in [0,\infty)$, set

$$\mathrm{v}_{\psi,T}(f) := \sup_\kappa \sum_{i=1}^n \psi\left(|f(t_i) - f(t_{i-1})|\right),$$

where κ ranges over all partitions $0 = t_0 \le t_1 \le \dots \le t_n = T$ of $[0,T]$ (cf. (5)). See [1] for a much more explicit expression for $\mathrm{v}_{\psi,T}(f)$.

Corollary 1. *For almost all ω for every $T \in [0,\infty)$*

$$\omega \textit{ is constant on } [0,T] \textit{ or } \mathrm{v}_{\psi,T}(\omega) \in (0,\infty). \tag{16}$$

Proof. First let us check that under the Wiener measure (16) holds for all T for almost all ω. It is sufficient to consider only rational T. Therefore, it is sufficient to consider a fixed rational T. And for a fixed T this follows from what Taylor proved in [18].

In view of Theorem 3 it suffices to check that the complement of the event (16) is space/time-superinvariant. It is sufficient to check (12), where E is the complement of (16), for $c = 0$. In other words, it is sufficient to check that $\omega^{(0,f)} = \omega \circ f$ satisfies (16) whenever ω satisfies (16). It remains to notice that $\mathrm{v}_{\psi,T}(\omega) = \mathrm{v}_{\psi,T'}(\omega \circ f)$, where $T' \in f^{-1}(T)$. □

Corollary 1 immediately implies that $\mathrm{vi}(\omega) = 2$ and $\mathrm{v}_2(\omega) = \infty$ almost surely, as claimed above.

Open problem. Can Corollary 1 be partially extended to càdlàg functions to say that $\mathrm{v}_{\psi,T}(\omega) < \infty$ a.s.?

The quantity $\mathrm{v}_{\psi,T}(f)$ is not nearly as fundamental as the following quantity introduced by Taylor [18]: for $f : [0,\infty) \to \mathbb{R}$ and $T \in [0,\infty)$, set

$$\mathrm{w}_T(f) := \lim_{\delta \to 0} \sup_{\kappa \in K_\delta[0,T]} \sum_{i=1}^{n_\kappa} \psi\left(|\omega(t_i) - \omega(t_{i-1})|\right), \tag{17}$$

where $K_\delta[0,T]$ is the set of all partitions $0 = t_0 \le \cdots \le t_{n_\kappa} = T$ of $[0,T]$ whose mesh is less than δ: $\max_i(t_i - t_{i-1}) < \delta$. Notice that the expression after the $\lim_{\delta\to0}$ in (17) is increasing in δ; therefore, $\mathrm{w}_T(f) \le \mathrm{v}_{\psi,T}(f)$.

Corollary 1 can be restated in terms of w:

Corollary 2. *For almost all ω for every $T \in [0,\infty)$*

$$\omega \text{ is constant on } [0,T] \text{ or } \mathrm{w}_T(\omega) \in (0,\infty). \tag{18}$$

Corollary 2 follows from this lemma:

Lemma 3. *For all $\omega \in \Omega$, $\mathrm{w}_T(\omega) < \infty$ if and only if $\mathrm{v}_{\psi,T}(\omega) < \infty$.*

Proof. It suffices to prove the part "only if". Let $\mathrm{w}_T(\omega) < \infty$ but $\mathrm{v}_{\psi,T}(\omega) = \infty$. Take any $\delta > 0$ such that the expression after the $\lim_{\delta\to0}$ in (17) is finite; without loss of generality let $\delta = T/N$ for some $N \in \{1, 2, \dots\}$. Let A be the value of this expression. Take any $C \ge \sup|\omega|$ and any partition $0 = t_0 \le t_1 \le \cdots \le t_n = T$ satisfying

$$\sum_{i=1}^{n} \psi\left(|\omega(t_i) - \omega(t_{i-1})|\right) > A + N \sup_{u \in [0,2C]} \psi(u).$$

Adding to this partition the points kT/N, $k = 1, 2, \dots, N-1$, we obtain a partition in $K_\delta[0,T]$ for which the expression after the $\lim_{\delta\to0}$ in (17) is greater than A. □

The value $\mathrm{w}_T(\omega)$ defined by (17) can be interpreted as the quadratic variation of the price path ω over the time interval $[0,T]$. Another non-stochastic definition of quadratic variation serves in [20] as the basis for the proof of Theorem 3 (informally, quadratic variation defines the time change transforming the price path into Brownian motion). The definition given in [20] is quite different from (17) and resembles Föllmer's [9] definition; in particular, the definition from [20] can be used to define the notion of stochastic integral w.r. to ω satisfying Itô's formula (cf. the theorem in [9]; this theorem is generalized in [12]).

The case of no borrowing and no short-selling

From this point on we consider the case of a finite horizon T. The definitions in this subsection are applicable both to the framework of Section 2 (where Ω is the set of positive càdlàg functions on $[0,T]$) and to the framework of this section (where Ω is the set of continuous functions on $[0,T]$).

We have only considered positive capital processes S_t. However, even positive capital processes may involve borrowing money or short-selling the security: at each time, S_t is the price of a portfolio containing some amounts of security and cash; the total value of the portfolio is positive but in principle its components can be strictly negative. In this section we consider the market where neither borrowing nor short-selling are allowed. Such markets have been considered by, e.g., Cover [3] and de Rooij and Koolen [5].

We can say that a pair (G,c), where G is a simple trading strategy and $c \in \mathbb{R}$ (the initial capital), is *prudent* if the corresponding capital process $\mathcal{K}_t^{G,c}$ is positive; in this note we have considered only such (G,c). As before, the components of G will be denoted τ_n (the stopping times) and h_n (the positions), and we imagine a trader with initial capital c who follows G. For $t \in [0,T]$ and $\omega \in \Omega$, set $h_t(\omega) := h_n(\omega)$, where n is the unique number satisfying $t \in (\tau_n, \tau_{n+1}]$; intuitively, h_t is the trader's position at time t. The amount of *cash* in the trader's portfolio at time t is defined to be $\mathcal{K}_t^{G,c}(\omega) - h_t(\omega)\omega(t)$. Let us say that (G,c) is *super-prudent* if, for all ω and t, we have $h_t(\omega) \ge 0$ (the condition of *no short-selling*) and $\mathcal{K}_t^{G,c}(\omega) - h_t(\omega)\omega(t) \ge 0$ (the condition of *no borrowing*).

In the framework of Section 2, where Ω is the set of positive càdlàg functions on $[0,T]$, all prudent pairs (G,c) are automatically strictly prudent. Indeed, let (G,c) be a prudent pair. If the condition of no short-selling is violated and $h_t(\omega) < 0$, we can make $\mathcal{K}_t^{G,c}(\omega) < 0$ by modifying ω over $[t,T]$ and making $\omega(t)$ sufficiently large. (Intuitively, short-selling is risky when the price path can jump since there is no upper limit on the security's price.) If the condition of no borrowing is violated and $\mathcal{K}_t^{G,c}(\omega) - h_t(\omega)\omega(t) < 0$, we can make $\mathcal{K}_t^{G,c}(\omega) < 0$ by modifying ω over $[t,T]$ and setting $\omega(t) := 0$. (Intuitively, borrowing is risky when the price path can jump since the price can drop to zero at any time.) We will see shortly that in the framework of this section, where Ω is the set of continuous functions on $[0,T]$, there is a big difference between prudent and strictly prudent pairs (G,c).

By a *super-prudent capital process* we will mean a process S that can be represented in the form (3) where all pairs (G_m, c_m) are required to be super-prudent and the positive series $\sum_{m=1}^{\infty} c_m$ is required to converge. This definition is applicable to the frameworks of both Section 2 and Section 3.

Let E be a set of positive continuous functions on $[0,T]$. Since $E \subseteq \Omega$ in the frameworks of both Section 2 and this section, *a priori* there are at least four natural definitions of the upper probability $\overline{\mathbb{P}}(E)$:

- $\overline{\mathbb{P}}_1(E)$ is the upper probability (8), exactly as it is defined there; namely, S ranges over the positive capital processes defined on the space Ω of all positive càdlàg functions on $[0,T]$;
- $\overline{\mathbb{P}}_2(E)$ is the upper probability (8) with S ranging over the super-prudent capital processes defined on the space Ω of all positive càdlàg functions on $[0,T]$;
- $\overline{\mathbb{P}}_3(E)$ is the upper probability (8) with S ranging over the positive capital processes defined on $\Omega = C[0,T]$;
- $\overline{\mathbb{P}}_4(E)$ is the upper probability (8) with S ranging over the super-prudent capital processes defined on $\Omega = C[0,T]$.

In fact, most of these definitions are equivalent:

Proposition 3. *For any set E of positive continuous functions on $[0,T]$,*

$$\overline{\mathbb{P}}_1(E) = \overline{\mathbb{P}}_2(E) = \overline{\mathbb{P}}_4(E).$$

There exists a set E of positive continuous functions on $[0,T]$ such that

$$\overline{\mathbb{P}}_1(E) = \overline{\mathbb{P}}_2(E) = \overline{\mathbb{P}}_4(E) \neq \overline{\mathbb{P}}_3(E).$$

Proof. The equality $\overline{\mathbb{P}}_1(E) = \overline{\mathbb{P}}_2(E)$ has already been demonstrated. The equality $\overline{\mathbb{P}}_2(E) = \overline{\mathbb{P}}_4(E)$ is obvious. Let us check that the inequality $\overline{\mathbb{P}}_1(E) > \overline{\mathbb{P}}_3(E)$ holds for some E. Let E be the set of all positive $\omega \in C[0,T]$ such that $\mathrm{vi}(\omega) \in (0,2)$. According to Theorem 2, $\overline{\mathbb{P}}_3(E) = 0$. And according to Proposition 2, $\overline{\mathbb{P}}_1(E) = 1$: there are even individual elements $\omega \in E$ for which $\overline{\mathbb{P}}_1(\{\omega\})$ is arbitrarily close to 1. □

4 Right-continuous price paths

The time interval is now again finite, $[0,T]$. We have already considered two choices for the set Ω of allowed price paths: $C[0,T]$ in Section 3 and the positive functions in $D[0,T]$ in Section 2. In one case $\mathcal{F}_t$ was generated simply by the projections $\omega \mapsto \omega(s)$, $s \le t$, and in the other case we applied, additionally, universal completion. In this section we assume that each price path ω is positive and has limits on the right, so that Ω is the set of all positive right-continuous functions $\omega : [0,T] \to \mathbb{R}$. Right-continuity is a natural relaxation of continuity that agrees with the direction of time: for each t, ω will not deviate much from $\omega(t)$ immediately after t. The choice of $\mathcal{F}_t$ is now much more natural than before: each σ-algebra $\mathcal{F}_t$ consists of all "cylinder sets", i.e., all sets $E \subseteq \Omega$ such that

$$\left(\omega \in E, \omega' \in \Omega, \omega|_{[0,t]} = \omega'|_{[0,t]}\right) \Longrightarrow \omega' \in E.$$

(Potential difficulties with this definition will be discussed later.) When defining $\overline{\mathbb{P}}(E)$ for $E \subseteq \Omega$, the $\liminf_{t\to\infty} S_t(\omega)$ in (8) is replaced by $S_T(\omega)$. Otherwise, all definitions are as before.

As an example of using our new definitions, we can state the following simple result (a version of [7], VI.3(2)).

Proposition 4. *Almost surely, the price path ω is càdlàg.*

Proof. It suffices to prove that the number of upcrossings of any open interval (a, b) with rational endpoints is finite almost surely ([6], Theorem IV.22). Fix such a and b. We can now follow the proof of Lemma 1; under the current definitions it is obvious that the hitting time of a closed interval is a stopping time. □

I know that the reader wants an assurance that our definitions are "free of contradiction": the σ-algebras of this section just appear too big. There is no formal contradiction, like the one we get assuming the existence of an extension of the Lebesgue measure on $[0, 1]$ to the power set of $[0, 1]$ (without rejecting the axiom of choice). But there is a danger that results such as Proposition 4 are vacuous. For example, is it possible that the trader has a strategy making him infinitely rich at time T no matter what ω crops up? It is easy to see that such a strategy does not exist: the initial capital will never increase if ω is constant. The next question is: is there a strategy that makes the trader infinitely rich when ω is not constant? Proposition 2 shows that this is not true. The following simple result, whose method of proof is borrowed from [4], strengthens the negative answer:

Lemma 4. *For any positive capital process S there exists a non-constant càdlàg $\omega \in \Omega$ such that $S_T(\omega) \le S_0$.*

Proof. Consider any representation of S in the form (3). Let (τ_n^m) and (h_n^m) be the stopping times and functions involved in the definition of G_m. The set of all stopping times τ_n^m is countable. Choose any $t \in (0, T)$ that is different from all $\tau_n^m(1)$, where 1 stands for the element of Ω that is identically equal to 1. For each m, let $n(m)$ be the largest integer such that $\tau_{n(m)}^m < t$ (with τ_0^m understood to be 0). Now we can define ω by the requirements that it should be equal to 1 in the interval $[0, t)$, be constant in the interval $[t, T]$, and satisfy

$$\omega(t) - \omega(t-) = \begin{cases} 1 & \text{if } \sum_m h_{n(m)}(1) \le 0 \\ -1 & \text{if } \sum_m h_{n(m)}(1) > 0. \end{cases} \quad \square$$

The step of Lemma 4 can be repeated more than once, which allows the market to choose from among a lot of piecewise constant functions ω without allowing the trader to increase his capital; and Proposition 2 allows the market to choose from among many continuous functions. However, the problem remains, and is especially acute in the framework of Section 3; e.g., the answer to the following question is unknown:

Open problem. Let Ω be the set $C[0,T]$ of all continuous functions $\omega : [0,T] \to \mathbb{R}$. Let E be the set of all non-constant functions in Ω. Is it true that $\overline{\mathbb{P}}(E) = 1$ (or at least $\overline{\mathbb{P}}(E) > 0$)?

The answer appears to be an obvious "yes", but after Banach–Tarski [21] we want a proof.

The analogue of the last problem when Ω is the set of all positive functions in $D[0,T]$ is answered by Proposition 2: according to (9), there are non-constant continuous ω with $\overline{\mathbb{P}}(\{\omega\})$ arbitrarily close to 1. However, the following question is open:

Open problem. Let Ω be the set of all positive càdlàg (or positive right-continuous) functions $\omega : [0,T] \to \mathbb{R}$. Let E be the set of all $\omega \in \Omega$ satisfying $\mathrm{vi}(\omega) = 2$. Is it true that $\overline{\mathbb{P}}(E) = 1$ (or at least $\overline{\mathbb{P}}(E) > 0$)?

At this point it is natural to show that we do not have similar problems for the definitions of Sections 2 and 3. Let $X_t : \Omega \to \mathbb{R}$ be the projection $X_t(\omega) := \omega(t)$, where Ω is defined as in either of these two sections.

Proposition 5. *Let X_t be a martingale w.r. to a probability measure P on $(\Omega, \mathcal{F}_T)$ and the filtration $(\mathcal{F}_t)$ (under the definitions of Section 2 or Section 3). If $E \in \mathcal{F}_T$ satisfies $P(E) = 1$, then $\overline{\mathbb{P}}(E) = 1$.*

Proof. Under P, any positive simple capital process becomes a positive càdlàg local martingale, since by the optional sampling theorem, every partial sum in (2) becomes a càdlàg martingale. Every positive local martingale is a supermartingale, and so the partial sums corresponding to a given positive capital process (3) are positive càdlàg supermartingales. Therefore, the existence of a positive capital process increasing its value between time 0 and T by more than a strictly positive constant for all $\omega \in \Omega$ would contradict the maximal inequality for positive càdlàg supermartingales (as applied to the partial sums). □

Proposition 5 shows that the results of Sections 2 and 3 are applicable to the typical paths of numerous stochastic processes, including Brownian motion, which is continuous and has typical paths ω satisfying $\mathrm{vi}(\omega) = 2$.

In principle, adopting the definitions of this section might lead to very different properties of typical price paths in efficient markets from what we are accustomed to. But even if this is correct, I believe that these properties deserve to be studied, simply because of the mathematical and intuitive simplicity of these definitions.

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