Discrete dynamic hedging without probability

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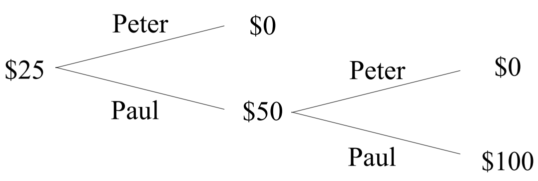

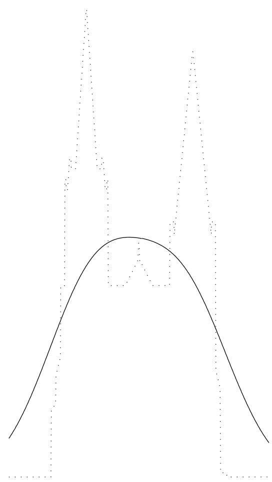

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Abstract

Even if the price of a security is not governed by a probability measure, a European option in the security can be hedged in discrete time by trading in the security and an instrument that pays its variance. A non-probabilistic bound on the error of the hedging is given.

Contents

1 Introduction

In [12], Vladimir Vovk and I demonstrated that an option $\mathcal{U}$ on a security $\mathcal{S}$ can be hedged dynamically in discrete time if the market prices an instrument, say $\mathcal{D}$, that pays directly the variance of the underlying security. This article reviews that demonstration, discusses its implications for risk management, and asks some questions about the practicality of an exchange for trading in $\mathcal{D}$.

Hedging in discrete time cannot be perfect. But the demonstration reviewed in this article establishes that its error can be bounded. No probabilities are involved in the bound.

Most theories of option pricing assume that the price of $\mathcal{S}$ follows some probability distribution. This is not assumed here. Instead, it is assumed that the price of $\mathcal{S}$ scales approximately as $\sqrt{\mathbf{d}t}$, while the price of $\mathcal{D}$ scales approximately as $\mathbf{d}t$. What it means for a time series to scale approximately as $(\mathbf{d}t)^{1/q}$ is explained in §2.

Often one also assumes a constant interest rate. In order to avoid the error introduced by this assumption, this article proposes that hedging be financed not with short-term money but with trades in a zero-coupon risk-free bond. If this proposal is adopted, it becomes mathematically convenient to measure prices and capital in units of the bond, so that the payment or receipt of interest is no longer explicit. The error bound demonstrated in [12], which is reviewed and explained heuristically in §3, is formulated in these terms.

The sensitivity of the value of a portfolio of options to the two main risk factors, changes in the prices of $\mathcal{S}$ and $\mathcal{D}$, is considered in §4. The partial derivative of the value with respect to the price of $\mathcal{S}$ can be called Delta, as in the established theory. The partial derivative with respect to the price of $\mathcal{D}$ is a new "Greek", which needs a new name; the proposed name is "Upsilon".

In §5 the analysis is restated in dollar terms, and the special case of a constant interest rate is reconsidered.

In §6, some practical questions are raised. Is it feasible to organize an exchange that allows ready trading in variance instruments of the type $\mathcal{D}$ and in the equivalent of zero-coupon risk-free bonds with maturities matching those of commonly marketed options? What further costs or approximation errors would be introduced by trying to do so?

There are two appendices. Appendix A gives a self-contained proof of the bound stated in §3. Appendix B relates the ideas discussed here to other work.

2 The variation spectrum

Let us now consider the notion of scaling relative to $\mathbf{d}t$ in discrete time.

Given a sequence $R_0, R_1, \ldots, R_N$ of real numbers and a positive real number p, set

$$\mathbf{var}_R(p) := \sum_{n=0}^{N-1} |\mathbf{d}R_n|^p ,$$

where $\mathbf{d}R_n := R_{n+1} - R_n$. The function $\mathbf{var}_R$ is the *variation spectrum* for $R_0, R_1, \ldots, R_N$.

Suppose $R_0, R_1, \ldots, R_N$ are successive values of a quantity $\mathcal{R}$, observed at evenly spaced times over the interval $[0, T]$, so that the time $\mathbf{d}t$ between R_n and R_{n+1} equals T/N. Suppose further that N is large, $\mathbf{d}t$ is small, and the $\mathbf{d}R_n$ have the same order magnitude, on average, as $(\mathbf{d}t)^{1/q}$. Then $\mathbf{var}_R(p)$ will have about the same order of magnitude as

$$\sum_{n=0}^{N-1} |\mathbf{d}t|^{p/q} = N^{(q-p)/q} T^{p/q}.$$

So if we hold T constant and increase N (thereby making $\mathbf{d}t$ smaller), $\mathbf{var}_R(p)$ will behave like $N^{(q-p)/q}$. If $p > q$, $\mathbf{var}_R(p)$ tends to zero as N grows; if $p < q$, it tends to infinity.

This article applies this general picture to $\mathcal{S}$ and $\mathcal{D}$ roughly as follows:

1. $\mathcal{S}$'s scaling approximately as $\sqrt{\mathbf{d}t}$ is interpreted to mean that $\mathbf{var}_S(p) \ll 1$ for p somewhat greater than 2.

2. $\mathcal{D}$'s scaling approximately as $\mathbf{d}t$ is interpreted to mean that $\mathbf{var}_D(p) \ll 1$ for p somewhat greater than 1.

The smaller $\mathbf{var}_S(p)$ and $\mathbf{var}_D(p)$ are for the relevant values of p, the smaller the error of the hedging strategy described in §3.

The rough statement of the preceding paragraph needs to be emended in two ways:

1. In the case of the underlying security $\mathcal{S}$, it is not the the variation spectrum $\mathbf{var}_S$, but the *relative variation spectrum* $\mathbf{var}_S^{\mathrm{rel}}$, given by

$$\mathbf{var}_S^{\mathrm{rel}}(p) := \sum_{n=0}^{N-1} \left| \frac{\mathbf{d}S_n}{S_n} \right|^p,$$

   that comes into play. It is dimensionless and invariant with respect to the unit in which S_n is measured, but it has the same asymptotic behavior as $\mathbf{var}_S(p)$.

2. There is a relation between the p for which $\mathbf{var}_S^{\mathrm{rel}}(p) \ll 1$ is needed and the p for which $\mathbf{var}_D(p) \ll 1$ is needed. We need $\mathbf{var}_S^{\mathrm{rel}}(2+\gamma) \ll 1$ and $\mathbf{var}_D(2-\gamma) \ll 1$ for some $\gamma \in (0,1)$. Any such γ will do, but it is convenient to consider $2/3$, because $\mathbf{var}_S^{\mathrm{rel}}(2+2/3)$ and $\mathbf{var}_D(2-2/3)$ are both of order $N^{-1/3}$. Empirical evidence ([12], §10.4) suggests that $\mathbf{var}_S^{\mathrm{rel}}(2+2/3)$ and $\mathbf{var}_D(2-2/3)$ will be reasonably small for highly liquid securities when $\mathbf{d}t$ is one week or one day.

3 Black-Scholes pricing in terms of a bond

When the variance instrument $\mathcal{D}$ is available, as we will now see, we can replicate a well behaved European option $\mathcal{U}$ with the same maturity as $\mathcal{D}$ by trading in $\mathcal{S}$ and $\mathcal{D}$. The cost of the replication, and hence the price of $\mathcal{U}$, is given by the Black-Scholes formula with the theoretical variance $T\sigma^2$ replaced by the initial market price of $\mathcal{D}$. There is no theoretical variance, because the price process for $\mathcal{S}$ is not stochastic.

As already mentioned, the discrete-time theory developed here requires that trades in $\mathcal{S}$ and $\mathcal{D}$ be financed by trades in a zero-coupon bond with the same maturity as $\mathcal{U}$ and $\mathcal{D}$. In this section, prices and capital are measured in units of this bond. The results are translated into dollar terms in §5.

The theory of this section applies to American as well as European options; see Chapter 13 of [12]. But for simplicity and clarity, the exposition here is limited to European options.

It is initially assumed that $\mathcal{S}$ does not pay dividends. The case where $\mathcal{S}$ does pay dividends is discussed in §3.4.

3.1 The protocol

Let us now consider the capital process for an investor who trades freely during N periods in $\mathcal{S}$, $\mathcal{D}$, and a zero-coupon risk-free bond with the same maturity as $\mathcal{D}$. In this idealized picture, the investor can go long or short in these securities in any amount, with no transaction costs.

Let S_0 and D_0 be the market prices of $\mathcal{S}$ and $\mathcal{D}$, respectively, at the start of trading, and let S_n and D_n be their prices at the end of the nth period and the beginning of the $(n+1)$st. These prices are in units of the bond.

At end of the nth period, $\mathcal{D}$ pays as a dividend a number of units of the bond equal to the square of $\mathcal{S}$'s return over that period, $(\mathbf{d}S_{n-1}/S_{n-1})^2$. Over the course of the N periods it pays a total of

$$\sum_{n=0}^{N-1} \left(\frac{\mathbf{d}S_n}{S_n} \right)^2 .$$

The dividends are the only source of value for $\mathcal{D}$. So D_0 is the value the market initially places on the total dividend stream, and D_n is the value it places on the dividend stream that remains at the end of the nth period,

$$\sum_{i=n}^{N-1} \left(\frac{\mathbf{d}S_i}{S_i} \right)^2 .$$

At the end of the N trading periods, $\mathcal{D}$ is worthless; $D_N = 0$.

Were the market to believe that the price of $\mathcal{S}$ follows geometric Brownian motion with variance σ^2, it would set

$$D_n := \frac{N-n}{N} T\sigma^2. \tag{1}$$

The market knows not to expect geometric Brownian motion, and so (1) will not hold with any exactness. But it is reasonable to expect the series $D_0, D_1, \ldots, D_N$ to be smoother than the series $S_0, S_1, \ldots, S_N$, scaling as $\mathbf{d}t$ rather than as $\sqrt{\mathbf{d}t}$.

If the investor holds M_n units of $\mathcal{S}$ and V_n units of $\mathcal{D}$ during the nth period, his profit for the period will be

$$M_n \mathbf{d}S_{n-1} + V_n \left(\left(\frac{\mathbf{d}S_{n-1}}{S_{n-1}} \right)^2 + \mathbf{d}D_{n-1} \right) \tag{2}$$

units of the bond.

This picture is summarized by the following protocol for the game that determines the investor's capital process K.

DISCRETE BLACK-SCHOLES PROTOCOL
Parameters: N, $K_0 > 0$, $\delta \in (0,1)$
Players: Market, Investor
Protocol:
Market announces $S_0 > 0$ and $D_0 > 0$.
FOR $n = 1, \ldots, N$:
Investor announces $M_n \in \mathbb{R}$ and $V_n \in \mathbb{R}$.
Market announces $S_n > 0$ and $D_n \geq 0$.
$K_n := K_{n-1} + M_n \mathbf{d}S_{n-1} + V_n((\mathbf{d}S_{n-1}/S_{n-1})^2 + \mathbf{d}D_{n-1})$.
Additional Constraints on Market: Market must set $D_N = 0$ and must choose the other S_n and D_n so that

$$\mathbf{var}_S^{\text{rel}}(2+\gamma) < \delta \quad \text{and} \quad \mathbf{var}_D(2-\gamma) < \delta \tag{3}$$

for some $\gamma \in (0,1)$.

This protocol is not subject to arbitrage by Investor, for Market can keep Investor from making any money at all by setting

$$D_n := \frac{N-n}{N} D_0 \quad \text{and} \quad S_n = S_{n-1} \pm S_{n-1} \sqrt{\frac{D_0}{N}},$$

the sign being opposite that of M_n. If he also chooses $D_0 < (\delta^3 N)^{1/4}$, he will satisfy the constraint (3) with $\gamma = 2/3$.

It is worth emphasizing that the protocol does not require Market to move stochastically. Market can move however he wants; he can even play strategically against Investor. The only constraint on his moves is (3).

3.2 The Black-Scholes formula

A European option $\mathcal{U}$ with maturity T is defined by its payoff function, say U; it pays $U(S_N)$ at time $T = N\mathbf{d}t$. Suppose p is a real number, and ϵ is a positive real number. Let us say that the *price* of $\mathcal{U}$ at the outset of trading (in the situation where S_0 and D_0 have just been announced) is *p with accuracy ϵ*

if Investor has a strategy in the discrete Black-Scholes protocol guaranteeing, when his initial capital K_0 is p, that his final capital K_N will satisfy

$$|U(S_N) - K_N| \leq \epsilon$$

provided Market obeys (3).[1]

Proposition 1 *Suppose $U\colon (0,\infty) \to \mathbb{R}$ is log-Lipschitzian[2] with coefficient c. Then in the situation where S_0 and D_0 have just been announced, the price of $\mathcal{U}$ is*

$$\int U(S_0 e^z)\; \mathcal{N}_{-D_0/2,\, D_0}(dz) \tag{4}$$

with accuracy

$$40c\delta^{1/4}. \tag{5}$$

Here $\mathcal{N}_{\mu,\,\sigma^2}$ is the Gaussian distribution with mean μ and variance σ^2. Equation (4) reduces, in the case of a call or a put, to the usual Black-Scholes formula, except that the interest rate is set to zero and D_0 is substituted for $T\sigma^2$.

As explained in §2, the parameter δ in the protocol can be chosen reasonably small if trading is frequent. The bound $40c\delta^{1/4}$ will still be too large to be useful for $\mathcal{U}$ with too large a value of c. But as the proof of Proposition 1 will reveal, this bound can be tightened substantially for most options (see Appendix A.3).

3.3 Heuristic proof

To see that (4) is the approximate cost of replicating $\mathcal{U}$, consider the function

$$\overline{U}(S, D) := \int_{\mathbb{R}} U(Se^z)\; \mathcal{N}_{-D/2,\, D}(dz). \tag{6}$$

This function is defined for $S \in \mathbb{R}$ and $D \geq 0$. It is easily verified that

$$\frac{\partial \overline{U}}{\partial D} = \frac{1}{2} S^2 \frac{\partial^2 \overline{U}}{\partial S^2} \tag{7}$$

when $D > 0$. When $D = 0$, $\mathcal{N}_{-D/2,\, D}$ puts all its probability on $z = 0$, and so

$$\overline{U}(S, 0) = U(S). \tag{8}$$

The differential equation (7) is a variant on the Black-Scholes equation, and (6) is its solution satisfying the initial condition (8).

The value of $\mathcal{D}$ at the end of trading, D_N, being zero, the quantity $\overline{U}(S_N, D_N)$ is equal, by (8), to $\mathcal{U}$'s payoff. Proposition 1 says that $\overline{U}(S_0, D_0)$ is

[1] Because of the symmetry of Investor's move space (he can replace M_n by $-M_n$ and V_n by $-V_n$), this implies that Investor can also replicate $-\mathcal{U}$ to within ϵ starting with $-p$. So he can buy or sell $\mathcal{U}$ for p and suffer a loss of at most ϵ.

[2] A function $f\colon \mathbb{R} \to \mathbb{R}$ is *Lipschitzian* with coefficient c if $|f(x) - f(y)| \leq c|x - y|$ for all x and y in $\mathbb{R}$. A function $U : (0,\infty) \to \mathbb{R}$ is *log-Lipschitzian* with coefficient c if $|U(e^x) - U(e^y)| \leq c|x - y|$ for all x and y in $\mathbb{R}$.

$\mathcal{U}$'s approximate price at the outset of trading. In fact, $\overline{U}$ approximately prices $\mathcal{U}$ throughout. Right after Market announces S_n and D_n, the approximate price of $\mathcal{U}$ is

$$U_n = \overline{U}(S_n, D_n).$$

This is because if Investor has capital $K_0 = U_0$ at the outset, he can trade in $\mathcal{S}$ and $\mathcal{D}$ in such a way that his capital at the end of each period n is approximately U_n. To show that this is so, it is enough to show that Investor can always choose his moves M_n and V_n so that $\mathbf{d}K_{n-1} = \mathbf{d}U_{n-1}$. To this end, approximate $\mathbf{d}U_{n-1}$ by a Taylor's series:

$$\mathbf{d}U_{n-1} \approx \frac{\partial \overline{U}}{\partial S}\mathbf{d}S_{n-1} + \frac{\partial \overline{U}}{\partial D}\mathbf{d}D_{n-1} + \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial S^2}(\mathbf{d}S_{n-1})^2. \tag{9}$$

We can neglect higher order terms because of the constraint (3). Comparing (9) with $\mathbf{d}K_{n-1}$, which is given by (2), we see that the two will be equal if

$$M_n = \frac{\partial \overline{U}}{\partial S}(S_{n-1}, D_{n-1}), \tag{10}$$

$$V_n = \frac{\partial \overline{U}}{\partial D}(S_{n-1}, D_{n-1}), \tag{11}$$

and

$$\frac{V_n}{S_{n-1}^2} = \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial S^2}(S_{n-1}, D_{n-1}). \tag{12}$$

Equations (10) and (11) give the moves M_n and V_n Investor needs to make. If he does make the move V_n given by (11), then (12) will also hold, because $\overline{U}$ satisfies the partial differential equation (7).

To prove Proposition 1, it remains to show that the constraint (3) suffices to keep the total hedging error caused by the approximations (9) from exceeding $40c\delta^{1/4}$. This is done in Appendix A.

3.4 When $\mathcal{S}$ pays dividends

How should the preceding theory be modified when $\mathcal{S}$ pays dividends?

Suppose first we know in advance how S_N will be related to the final final value S_N^* of one share assuming its dividends are reinvested; we know a strictly monotonic function f such that $S_N^* = f(S_N)$. We might, for example, know there will be a continuous dividend payment at rate q, so that

$$S_N^* = e^{qT} S_N.$$

Then we can simply re-express the problem in terms of S^*; this means $\mathcal{D}$'s dividend should be $\mathcal{S}$'s total return, which is $(\mathbf{d}S^*/S^*)^2$, and the pricing formula (4) should be applied not to U but to the function U^*, where

$$U^*(x) := U(f^{-1}(x)),$$

so that $U^*(S_N^*) = U(S_N)$. This approach is appropriate when $\mathcal{S}$ is the currency issued by a foreign government with a risk-free interest rate that differs by a constant amount from the domestic rate.

A different approach is appropriate when one knows at the outset the dates and dollar amounts of the dividends. In this case, each share of $\mathcal{S}$ can be regarded as portfolio, consisting of a share that does not pay dividends and a portfolio of bonds that can, if desired, be hedged by taking an opposite position in risk-free bonds of the same maturities. The dividend for $\mathcal{D}$ should then be simply $\mathbf{d}S/S$, the percentage price change ignoring any dividends from $\mathcal{S}$, and the pricing formula (4) should be applied to U.

4 The Greeks

Let us turn now to the Greeks, which measure the sensitivity of the price of an option to various risk factors.

4.1 The Greeks for a single option

The partial derivatives (10) and (11) measure the sensitivities of $\mathcal{U}$'s price to the prices of $\mathcal{S}$ and $\mathcal{D}$. Let us call them Delta and Upsilon:

$$\Delta := \frac{\partial \overline{U}}{\partial S}, \quad \Upsilon := \frac{\partial \overline{U}}{\partial D}.$$

The Black-Scholes equation, (7), tells us that

$$\frac{\partial^2 \overline{U}}{\partial S^2} = \frac{2}{S^2}\Upsilon \tag{13}$$

So the Taylor's series (9) can be written in the form

$$\mathbf{d}U \approx \Delta \mathbf{d}S + \Upsilon \left(\mathbf{d}D + \left(\frac{\mathbf{d}S}{S} \right)^2 \right). \tag{14}$$

In the small period of time $\mathbf{d}t$, the instrument $\mathcal{D}$ pays out the dividend $(\mathbf{d}S/S)^2$, thus diminishing in value by this amount. In addition, Market changes its opinion about the value of $\mathcal{D}$'s future dividends by some amount, say $\mathbf{d}^*D$. The total change in $\mathcal{D}$'s value, $\mathbf{d}D$, is therefore

$$\mathbf{d}D = \mathbf{d}^*D - \left(\frac{\mathbf{d}S}{S} \right)^2.$$

So (14) can be written as

$$\mathbf{d}U \approx \Delta \mathbf{d}S + \Upsilon \mathbf{d}^*D. \tag{15}$$

This equation tells us the meaning of Δ and Υ: Δ is the sensitivity of $\mathcal{U}$'s price to changes in the price of $\mathcal{S}$, while Υ is the sensitivity of $\mathcal{U}$'s price to changes in the price of $\mathcal{S}$'s future variance.

The security $\mathcal{S}$ itself is a European option on $\mathcal{S}$ with the same maturity as $\mathcal{D}$; its Delta is one, while its Upsilon is zero. Similarly, the risk-free bond with the same maturity as $\mathcal{D}$ is a European option on $\mathcal{S}$; its Delta and Upsilon are both zero. The derivative security $\mathcal{D}$ is a non-European option, but its price is certainly a function of the price of $\mathcal{D}$ and the price of $\mathcal{S}$; differentiating with respect to these prices, we see that its Delta is zero and its Upsilon is one.

The two Greeks can also be defined for a portfolio $\mathcal{P}$ consisting of units of $\mathcal{D}$ and various European options on $\mathcal{S}$ with the same maturity as $\mathcal{D}$. Assuming that the options in $\mathcal{P}$ are indexed by a finite set $\mathcal{I}$, say $\mathcal{P} = \{\mathcal{U}^I\}_{I\in\mathcal{I}}$, and writing Δ^I and Υ^I for the Delta and Upsilon for $\mathcal{U}^I$, one sets

$$\text{total Delta for } \mathcal{P} := \sum_I \Delta^I;$$

$$\text{total Upsilon for } \mathcal{P} := \sum_I \Upsilon^I;$$

These definitions makes sense because the value of the portfolio is the sum of the values of the options in it, and the partial derivative of a sum is the sum of the partial derivatives.

It is instructive to compare this picture with that of the usual theory, where the value of the option is a function of the security price, its volatility, and time:

$$\overline{U}^*(S,\sigma,t) := \int_{\mathbb{R}} U(Se^z)\ \mathcal{N}_{-(T-t)\sigma^2/2,\,(T-t)\sigma^2}(dz).$$

As explained in textbooks (e.g., [9]), one monitors the value of an option or a portfolio of options by looking at several partial derivatives, including:

$$\texttt{Delta} := \frac{\partial \overline{U}^*}{\partial S}, \quad \texttt{Vega} := \frac{\partial \overline{U}^*}{\partial \sigma}, \quad \texttt{Theta} := \frac{\partial \overline{U}^*}{\partial t}, \quad \texttt{Gamma} := \frac{\partial^2 \overline{U}^*}{\partial S^2}.$$

Our Delta is analogous to `Delta`, and but our Upsilon combines, in some respects, the roles of `Vega`, `Theta`, and `Gamma`.

The usual theory considers both `Vega` and `Theta` because the theoretical variance σ^2 and the time t both enter into $(T-t)\sigma^2$, the total variance of $\mathcal{S}$'s price between time t and time T. In the present picture, there is no stochastic assumption and hence no theoretical variance, but the total remaining empirical variance is priced by the market, and Υ measures the sensitivity of $\mathcal{U}$'s price to it.

Equation (13) tells us that Υ also does `Gamma`'s job. By monitoring Upsilon, we also monitor not only the sensitivity of $\mathcal{U}$'s price to $\mathbf{d}^*D$, the change in the market's valuation of $\mathcal{S}$'s future variance, but also its second order sensitivity to $\mathcal{S}$'s immediate change—its sensitivity to $(\mathbf{d}S/S)^2$.

4.2 The Greeks for multiple maturities and securities

Consider now a finite set of securities, say $\mathcal{S}^J$ for J in a finite index set $\mathcal{J}$. Consider also a finite set of different maturities, say $\mathcal{T}$. Suppose that a version of our dividend-paying instrument, say $\mathcal{D}^{JT}$, is marketed for each $\mathcal{S}^J$ and each maturity $T \in \mathcal{T}$.

Suppose further that $\mathcal{P}^{JT}$, for each $J \in \mathcal{J}$ and $T \in \mathcal{T}$, is a portfolio consisting of units of $\mathcal{D}^{JT}$ and European options on $\mathcal{S}^J$ with maturity T (including perhaps shares of $\mathcal{S}^J$ and units of the risk-free bond with maturity T). Set

$$\Delta^{JT} := \text{ total Delta for } \mathcal{P}^{JT};$$

$$\Upsilon^{JT} := \text{ total Upsilon for } \mathcal{P}^{JT};$$

For each J, let $\mathcal{P}^{J\cdot}$ be the portfolio obtained by pooling the portfolios $\mathcal{P}^{JT}$ for that J and all $T \in \mathcal{T}$, and set

$$\Delta^{J\cdot} := \sum_{T \in \mathcal{T}} \Delta^{JT}.$$

This is the total number of shares of $\mathcal{S}^J$ that one needs to short in order to make $\mathcal{P}^{JT}$ Delta-neutral for all $T \in \mathcal{T}$. Because this total does not change if we move long or short positions in $\mathcal{S}^J$ that happen to be in these portfolios from one of them to another, we risk no confusion if we say this more simply: $\Delta^{J\cdot}$ is the number of shares of $\mathcal{S}^J$ that one needs to short in order to make $\mathcal{P}^{J\cdot}$ Delta-neutral.

To summarize: monitoring the Greeks for a portfolio of European options on many securities with many maturities means monitoring the total Delta for each security and the total Upsilon for each (security,maturity) pair.

5 Black-Scholes pricing in dollars

This section translates the picture into dollar terms. The translation is routine with respect to the investor's capital and the prices of the securities. But the instrument $\mathcal{D}$ poses novel issues, because the dollar value of its dividend is affected by changes in the value of the bond.

5.1 The protocol

The protocol in §3.1, which lays out how Investor can trade in $\mathcal{S}$ and $\mathcal{D}$ during N trading periods, does not mention interest, because the prices S_n and D_n and Investor's capital process K_n are all measured in units of a zero-coupon bond. But the bond's returns plays an important implicit role, because Investor's residual (positive or negative) capital during each period, the capital he is not holding in $\mathcal{S}$ or $\mathcal{D}$, remains in the bond.

To translate the story into dollar terms, we must say explicitly that the Investor is constantly (in discrete time) rebalancing a portfolio consisting of shares of $\mathcal{S}$, $\mathcal{D}$ and a zero-coupon bond $\mathcal{B}$ with maturity N. The bond is supposed to be risk-free in the usual sense—it will not default. We may assume that each unit of the bond pays \$1 at maturity.

Let us write b_n for the price of $\mathcal{B}$ in dollars at the end of the nth trading period, and let us also use lower-case letters for other dollar amounts:

$$k_n := b_n K_n, \quad s_n := b_n S_n, \quad d_n := b_n D_n, \quad u_n := b_n U_n.$$

Let us also set

$$r_n :== \frac{\mathbf{d} b_{n-1}}{b_{n-1}}.$$

This is the return earned by $\mathcal{B}$ during the nth trading period.

The dividend paid by $\mathcal{D}$ at the end of the nth trading period is

$$\left(\frac{\mathbf{d} S_{n-1}}{S_{n-1}}\right)^2$$

in units of $\mathcal{B}$, or

$$b_n \left(\frac{\mathbf{d} S_{n-1}}{S_{n-1}}\right)^2 = \frac{b_{n-1}^2}{b_n} \left(\frac{\mathbf{d} s_{n-1}}{s_{n-1}} - r_n\right)^2 \tag{16}$$

in dollars.

Typically $b_n \approx b_{n-1}$, and so (16) is approximately equal to

$$b_n \left(\frac{\mathbf{d} s_{n-1}}{s_{n-1}} - r_n\right)^2,$$

which is the dollar value of a number of units of $\mathcal{B}$ equal to the squared difference between $\mathcal{S}$'s and $\mathcal{B}$'s returns. This is another way of saying that $\mathcal{D}$'s dividend is the $\mathcal{S}$'s squared return relative to the bond.

The rule for updating Investor's capital in the discrete Black-Scholes protocol can be written in the form

$$K_n := (K_{n-1} - M_n S_{n-1} - V_n D_{n-1}) + (M_n S_n + V_n D_n) + V_n \left(\frac{\mathbf{d} S_{n-1}}{S_{n-1}}\right)^2.$$

In dollar terms, this becomes

$$k_n := (k_{n-1} - M_n s_{n-1} - V_n d_{n-1}) \frac{b_n}{b_{n-1}} + (M_n s_n + V_n d_n) + V_n \frac{b_{n-1}^2}{b_n} \left(\frac{\mathbf{d} s_{n-1}}{s_{n-1}} - r_n\right)^2.$$

The quantity $k_{n-1} - M_n s_{n-1} - V_n d_{n-1}$ is the dollar amount Investor holds in $\mathcal{B}$ during the nth trading period; it is multiplied by b_n/b_{n-1} or $1 + r_n$ to account for the change in the value of $\mathcal{B}$ from the beginning to the end of the period. The second term, $M_n s_n + V_n d_n$, is the dollar amount Investor receives or pays when he liquidates his positions in $\mathcal{S}$ and $\mathcal{D}$ at the end of the period, and the third term represents his dividends from $\mathcal{D}$.

So the discrete Black-Scholes protocol looks like this in dollar terms:

Discrete Black-Scholes protocol in dollar terms

Parameters: N, $K_0 > 0$, $\delta \in (0,1)$
Players: Market, Investor
Protocol:
Market announces $b_0 > 0$, $s_0 > 0$, and $d_0 > 0$.
FOR $n = 1, \ldots, N$:
Investor announces $M_n \in \mathbb{R}$ and $V_n \in \mathbb{R}$.
Market announces $b_n > 0$, $s_n > 0$, and $d_n \geq 0$.

$$k_n := (k_{n-1} - M_n s_{n-1} - V_n d_{n-1}) \tfrac{b_n}{b_{n-1}} + (M_n s_n + V_n d_n) + V_n \tfrac{b_{n-1}^2}{b_n} \left(\tfrac{\mathbf{d} s_{n-1}}{s_{n-1}} - \tfrac{\mathbf{d} b_{n-1}}{b_{n-1}} \right)^2. \quad (16)$$

Additional Constraints on Market: Market must set $b_N = 1$ and $D_N = 0$ and must choose the other b_n, s_n, and d_n so that $\mathbf{var}_S^{\mathrm{rel}}(2+\gamma) < \delta$ and $\mathbf{var}_D(2-\gamma) < \delta$ for some $\gamma \in (0,1)$, where $S_n := s_n/b_n$ and $D_n := d_n/b_n$.

5.2 The Black-Scholes formula and the Greeks

Because $b_N = 1$, the shift from pricing in units of $\mathcal{B}$ to pricing in dollars makes no difference in how we describe a European option $\mathcal{U}$ with maturity at the end of the Nth period; its payoff function U gives its payoff in units of $\mathcal{B}$, which is the same as its payoff in dollars, and this can be described as $U(S_N)$ or $U(s_N)$; the two are the same.

Proposition 1 tells us that $\mathcal{U}$'s initial price in units of $\mathcal{B}$ is

$$U_0 = \overline{U}(S_0, D_0).$$

with accuracy $40c\delta^{1/4}$. So its initial price in dollars is

$$u_0 = b_0 \overline{U}(S_0, D_0) = b_0 \overline{U}\left(\frac{s_0}{b_0}, \frac{d_0}{b_0} \right).$$

with accuracy $b_0 40c\delta^{1/4}$. More generally, the price in dollars just after b_n, s_n, and d_n are announced is approximately

$$u_n = b_n \overline{U}\left(\frac{s_n}{b_n}, \frac{d_n}{b_n} \right) = b_n \int U(\frac{s_n}{b_n} e^z)\, \mathcal{N}_{-d_n/2b_n,\, d_n/b_n}(dz). \quad (17)$$

This is the Black-Scholes equation in dollar terms.

The Greeks Δ, Υ, and Γ, defined in §4.1, can be obtained by differentiating the function $\overline{u}$ defined by

$$\overline{u}(s,d,b) := b\overline{U}\left(\frac{s}{b}, \frac{d}{b} \right) = b_n \int U\left(\frac{s}{b} e^z \right) \mathcal{N}_{-d/2b,\, d/b}(dz).$$

Indeed,

$$\Delta = \frac{\partial \overline{u}}{\partial s}(s,d,b),$$

$$\Upsilon = \frac{\partial \overline{u}}{\partial d}(s,d,b),$$

and

$$\Gamma = b\frac{\partial^2 \overline{u}}{\partial s^2}(s,d,b)\,.$$

Moreover, we can use the derivative with respect to b to tell us how many units of the bond to hold, for

$$\frac{\partial \overline{u}}{\partial b}(s,d,b) = \overline{U}(S,D) - \Delta S - \Upsilon D,$$

which is the part of Investor's capital $\overline{U}(S,D)$ that remains in the bond when he holds Δ units of $\mathcal{S}$ and Υ units of $\mathcal{D}$.

In order to understand why Delta and Upsilon come out the same when we measure value in dollars as when we measure value in units of a bond, we must remember that the Greeks are ratios of prices and hence are dimensionless. Delta, for example, is the ratio

$$\frac{\text{change in the value of } \mathcal{U}}{\text{change in the value of a share of } \mathcal{S}},$$

and this ratio does not depend on whether value is being measured in units of a bond, dollars, or units of some other currency.

5.3 The picture with a constant interest rate

The usual Black-Scholes theory assumes that the interest rate is constant. So it is instructive to consider how the theory of this article looks under this assumption.

If we write r for a constant short-term interest rate with continuous compounding, then the corresponding notional bond paying \$1 at maturity N has the price

$$b_n := e^{-r(T-t)} \tag{18}$$

at the end of the nth trading period, where $t = n\mathbf{d}t$.

If we define the instrument $\mathcal{D}$ using this notional bond, then $\mathcal{D}$'s dollar dividend at the end of the nth period will be

$$e^{-r(N-n+2)\mathbf{d}t}\left(\frac{\mathbf{d}s_{n-1}}{s_{n-1}} - \frac{\ln(1+r)}{\mathbf{d}t}\right)^2. \tag{19}$$

Our formula for u_n, Equation (17), becomes

$$\begin{aligned} u_n &= e^{-r(T-t)}\overline{U}(s_n e^{r(T-t)}, d_n e^{r(T-t)}) \\ &= e^{-r(T-t)}\int U(s_n e^{z+r(T-t)})\,\mathcal{N}_{-d_n e^{r(T-t)}/2,\, d_n e^{r(T-t)}}(dz). \end{aligned}$$

So we can discuss hedging in terms of the function $\overline{u}^\dagger$ defined by

$$\overline{u}^\dagger(s,d,t,r) := e^{-r(T-t)}\int U(se^{z+r(T-t)})\,\mathcal{N}_{-de^{r(T-t)}/2,\, de^{r(T-t)}}(dz).$$

Our hedges in $\mathcal{S}$ and $\mathcal{D}$ can of course be found from $\overline{u}^\dagger$:

$$\Delta = \frac{\partial \overline{u}^\dagger}{\partial s}(s,d,t,r) \quad \text{and} \quad \Upsilon = \frac{\partial \overline{u}^\dagger}{\partial s}(s,d,t,r). \tag{20}$$

We might also use $\overline{u}^\dagger$ to examine the sensitivity of $\mathcal{U}$'s value to changes in the interest rate r and the time t.

Because r is not really constant, Proposition 1's bound on the error of replication will not be valid when the hedges (20) are financed with a money market account. It would be helpful to have bounds on the additional error introduced by variation in r; presumably it is substantial.

Variation in the interest rate also invalidates the usual Black-Scholes theory. Here, however, we encounter a question that does not arise in the usual theory. Were we to establish a market in $\mathcal{D}$ on the assumption of a constant interest rate, should we define its dividend (19) for all N periods using the short rate r in effect at the outset? Or should we acknowledge reality and change r in the formula as the short rate changes? Presumably we should to the latter, always using the current short rate in the formula, because this will make subsequent hedging as effective as possible.

Dynamic Black-Scholes hedging is presently treated as a rather idealized and heuristic idea; theory and practice agree, for example, that dynamic hedging of a call or a put is not practical. On the theoretical side, we find that bounds on the hedging error are much too loose to be useful [3]. On the practical side, we see that calls and puts are actually priced by markets, and these are used to hedge other options statically as much as possible. The purpose of setting up a market in $\mathcal{D}$ would be to get a little more mileage out of dynamic hedging. Calls and puts would actually be hedged dynamically, to a substantial degree, from $\mathcal{S}$ and $\mathcal{D}$. In this context, error bounds such as the one in Proposition 1 take on real importance, and the additional error added by the assumption of a constant interest rate is no longer tolerable. Traders in the proposed market could not make do with money market accounts; they would really need a liquid market in bonds with maturities matching those of the options they want to hedge.

6 Questions

There is reason to think that exchange trading of the variance instrument $\mathcal{D}$ could substantially improve the efficiency of option markets. Existing option exchanges market a two dimensional array of calls and puts for each underlying security $\mathcal{S}$; one dimension is the strike price and the other the maturity. Marketing $\mathcal{D}$ means marketing only a one-dimensional array of instruments; we market $\mathcal{D}^T$ for a range of maturities T, and according to Proposition 1, this determines prices for calls and puts for all strikes, with no volatility smile. Thus a liquid market would be achieved with a much smaller overall level of trading activity.

But the project of creating an exchange in $\mathcal{D}$ raises questions. Who is in a position to do this? Under what circumstances will it be feasible for participants in such an exchange to finance their trades in $\mathcal{D}$ and its underlying security $\mathcal{S}$

using a risk-free bond rather than money market accounts? One purpose of this paper is to stimulate thought about these questions by those most knowledgeable about option and bond markets.

6.1 How can an exchange in $\mathcal{D}$ be created?

Creating an exchange in $\mathcal{D}$ will require organizing a group of option traders and brokers who agree on a risk-free zero-coupon bond with which to define $\mathcal{D}$'s dividends and who can move in and out of such a bond with negligible transaction costs. This suggests that the exchange might best be organized by an institution strong enough that it itself can issue bonds considered risk free. It also suggests that the major participants in the exchange may tend to be traders who have other reasons for trading in bonds with the same maturities as the options being traded.

In the past few years, an increasingly vigorous over-the-counter market has developed in variance swaps for stock indices. When they were introduced, these instruments were priced and hedged statically using calls and puts, but their increasing liquidity has encouraged consideration of their use in pricing other options [4]. Could this increasingly liquid over-the-counter market be transformed into an an exchange?

Similar questions can be asked about the exchange the Chicago Board Options Exchange has organized in the volatility of the S&P 500 [5]. It presently trades in options and futures on VIX, a volatility index. Could the demand for futures on VIX be shifted to a demand for forwards on the variance of individual securities?

6.2 Should $\mathcal{D}$'s dividends be deferred?

Instead of paying the dividend $\mathbf{d}S_{n-1}/S_{n-1}$ at the end of the nth trading period, the instrument $\mathcal{D}$ could accumulate these dividends and pay them at the end of the N trading periods. It would then look much more like a variance swap.

In the theoretical picture, this seems to be a distinction without a difference. The dividend is denominated in units of a risk-free zero-coupon bond that matures at the end of the N trading periods, and Investor, it is assumed, can move his capital freely in and out of this bond, incurring no trading cost as he goes long or short in the bond in order to buy or sell $\mathcal{S}$ and $\mathcal{D}$. So there is no difference between giving Investor units of the bond and holding these units for him until they mature.

In practice, there must be some cost involved in trading in the bond, and it would be useful to discuss both how this can be minimized and how its impact on our error bound can be assessed.

6.3 Options on bonds?

Options on bonds are now usually priced by different methods than options on equity, because the different scope for variation in bond prices mandates different stochastic models. But the approach of this paper, which does not require a stochastic model, can be applied to options on bonds just as easily as to options on equity.

The volatility of bonds is now comparable to the volatility of equity [10], and the approach being considered here seems to point towards a greater integration of markets in the two types of volatility. The merits and demerits of such integration should be discussed.

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A Proof of Proposition 1

Proposition 1 appears in §10.5 of [12]. Unfortunately, its proof is spread over several chapters of [12]. The proof given here is self contained.

Before giving the proof, we review some mathematical tools it uses. After giving the proof, we explain how its techniques can be used to obtain tighter bounds for particular European options.

A.1 Mathematical prerequisites

Inequalities. The proof uses two well-known inequalities involving nonnegative sequences X_n and Y_n and nonnegative real numbers a and b. The first, Hölder's inequality, says that if $1/a + 1/b = 1$, then

$$\sum_{n=1}^{N} X_n Y_n \leq \left(\sum_{n=1}^{N} X_n^a\right)^{1/a} \left(\sum_{n=1}^{N} Y_n^b\right)^{1/b}. \tag{21}$$

The other, associated with the names Pringsheim and Jensen, says that if $a \leq b$, then

$$\left(\sum_{n=1}^{N} X_n^a\right)^{1/a} \geq \left(\sum_{n=1}^{N} X_n^b\right)^{1/b}. \tag{22}$$

For proofs, see, for example, [1, 8].

The inequality (22) can also be written in the form

$$\sum_{n=1}^{N} X_n^b \leq \left(\sum_{n=1}^{N} X_n^a\right)^{b/a}. \tag{23}$$

The Hölder and Pringsheim-Jensen inequalities together imply that (21) holds whenever $1/a + 1/b \geq 1$. In particular,

$$\sum_{n=1}^{N} X_n Y_n \leq \left(\sum_{n=1}^{N} X_n^{2+\gamma}\right)^{1/(2+\gamma)} \left(\sum_{n=1}^{N} Y_n^{2-\gamma}\right)^{1/(2-\gamma)}. \tag{24}$$

The gamma function. The gamma function is defined by

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt$$

for every positive real number x. Important values of the function are $\Gamma(1) = 0$ and $\Gamma(1/2) = \sqrt{\pi}$. These two values, together with the recursion $\Gamma(x+1) = x\Gamma(x)$, allow one to calculate $\Gamma(n)$ and $\Gamma(n/2)$ when n is a positive integer; we get $\Gamma(n) = n!$ for all n and

$$\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}, \quad \Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi}, \quad \Gamma(\frac{7}{2}) = \frac{15}{8}\sqrt{\pi}, \quad \text{etc.} \tag{25}$$

The proof requires evaluation of the integral $\int_0^\infty t^x e^{-t^2/2} dt$ for several small integer values of x. A simple change of variables shows that

$$\int_0^\infty t^x e^{-t^2/2} dt = 2^{(x-1)/2} \Gamma(\frac{x+1}{2}), \tag{26}$$

which combines with (25) to produce the needed evaluations.

Hermite polynomials. The Hermite polynomials used in the proof can be defined by

$$H_0(x) := 1 \quad \text{and} \quad H_1(x) := x, \tag{27}$$

together with the recursion

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x). \tag{28}$$

See, e.g., [14], Example II.11.1. In addition to H_0 and H_1, the proof also uses three more of these polynomials:

$$H_2(x) := x^2 - 1, \quad H_3(x) := x^3 - 3x, \quad \text{and} \quad H_4(x) := x^4 - 6x^2 + 3.$$

Using (27) and (28), one may prove by induction that

$$H_n'(x) = nH_{n-1}(x),$$

and it follows from this and (28) that

$$H_{n+1}(x) = xH_n(x) - H_n'(x). \tag{29}$$

The following lemma shows how these polynomials arise when we calculate derivatives of a function g obtained from Gaussian smoothing of another function f.

Lemma 1 *Suppose f is a c-Lipschitzian function, and set*

$$g(x) := \int f(x+z) \mathcal{N}_{0,\sigma^2}(dz).$$

Then

$$g^{(n)}(x) = \frac{1}{\sqrt{2\pi}\sigma^{n+1}} \int_{\mathbb{R}} e^{-(z-x)^2/(2\sigma^2)} H_n\left(\frac{z-x}{\sigma}\right) f(z) dz. \tag{30}$$

for $n = 0, 1, \ldots$.

The lemma can established by induction on n. The case $n = 0$ follows from the fact that $H_0(x) = 1$, and once we have (30) we can obtain the analogous relation for $n+1$ by differentiating (30) and using (29):

$$
\begin{aligned}
&g^{(n+1)}(x) \\
&= \frac{1}{\sqrt{2\pi}\sigma^{n+1}} \int_{\mathbb{R}} e^{-(z-x)^2/(2\sigma^2)} \left(\frac{z-x}{\sigma^2} H_n\left(\frac{z-x}{\sigma}\right) - \frac{1}{\sigma} H_n'\left(\frac{z-x}{\sigma}\right)\right) f(z) dz \\
&\qquad = \frac{1}{\sqrt{2\pi}\sigma^{n+2}} \int_{\mathbb{R}} e^{-(z-x)^2/(2\sigma^2)} H_{n+1}\left(\frac{z-x}{\sigma}\right) f(z) dz.
\end{aligned}
$$

A.2 The proof proper

The function $\overline{U}$ was defined by Equation (6). It is the only solution of the differential equation (7) satisfying the initial condition $\overline{U}(S,0) = U(S)$ and the polynomial growth condition (see [7], Appendix E). As explained in §3.3, Proposition 1 is proven by showing that if Investor begins with the capital $\overline{U}(S_0, D_0)$ and plays the strategy given by (10) and (11), then his capital will be close to $\overline{U}(S_n, D_n)$ at the end of each trading period n, finally arriving at the end of trading at a value close to $\mathcal{U}$'s payoff $\overline{U}(S_N, D_N) = \overline{U}(S_N, 0) = U(S_N)$.

The only assumption concerning U is that it is log-Lipschitzian with coefficient c. But let us first analyze the situation under the alternative assumption that U has at least four derivatives, all tending to zero fast enough as S increases that the quantities

$$
c_m := \sup_{S \in (0,\infty)} \left| S^m U^{(m)}(S) \right| \tag{31}
$$

are finite for $m = 2, 3, 4$.

Lemma 2 *Under the assumption just stated, $\overline{U}(S_0, D_0)$ is the price of $\mathcal{U}$ with accuracy*

$$
\delta \left(1.75 c_2 + 2.5 c_3 + 0.375 c_4\right). \tag{32}
$$

Proof First expand $\overline{U}$ in a Taylor's series at the point (S_n, D_n) as follows:

$$
\begin{aligned}
\mathbf{d}\overline{U}(S_n, D_n) = \frac{\partial \overline{U}}{\partial S}(S_n, D_n)\mathbf{d}S_n + \frac{\partial \overline{U}}{\partial D}(S_n, D_n)\mathbf{d}D_n \\
+ \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial S^2}(S_n', D_n')(\mathbf{d}S_n)^2 + \frac{\partial^2 \overline{U}}{\partial S \partial D}(S_n', D_n')\mathbf{d}S_n \mathbf{d}D_n \\
+ \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial D^2}(S_n', D_n')(\mathbf{d}D_n)^2
\end{aligned} \tag{33}
$$

for $n = 0, \ldots, N-1$, where (S_n', D_n') is a point strictly between (S_n, D_n) and (S_{n+1}, D_{n+1}). Then expand $\partial^2 \overline{U}/\partial S^2$ in a Taylor's series:

$$
\frac{\partial^2 \overline{U}}{\partial S^2}(S_n', D_n') = \frac{\partial^2 \overline{U}}{\partial S^2}(S_n, D_n) + \frac{\partial^3 \overline{U}}{\partial S^3}(S_n'', D_n'')\mathbf{d}S_n' + \frac{\partial^3 \overline{U}}{\partial D \partial S^2}(S_n'', D_n'')\mathbf{d}D_n', \tag{34}
$$

where (S''_n, D''_n) is a point strictly between (S_n, D_n) and (S'_n, D'_n). Since $\mathbf{d}S'_n = S'_n - S_n$ and $\mathbf{d}S_n = S_{n+1} - S_n$, we have $|\mathbf{d}S'_n| < |\mathbf{d}S_n|$; similarly, $|\mathbf{d}D'_n| < |\mathbf{d}D_n|$. Plugging (34) and (7) into (33) gives

$$\begin{aligned}\mathbf{d}\overline{U}(S_n, D_n) &= \frac{\partial \overline{U}}{\partial S}(S_n, D_n)\mathbf{d}S_n + \frac{\partial \overline{U}}{\partial D}(S_n, D_n)\left(\mathbf{d}D_n + \left(\frac{\mathbf{d}S_n}{S_n}\right)^2\right) \\ &+ \frac{1}{2}\frac{\partial^3 \overline{U}}{\partial S^3}(S''_n, D''_n)\mathbf{d}S'_n(\mathbf{d}S_n)^2 + \frac{1}{2}\frac{\partial^3 \overline{U}}{\partial S^2 \partial D}(S''_n, D''_n)\mathbf{d}D'_n(\mathbf{d}S_n)^2 \\ &+ \frac{\partial^2 \overline{U}}{\partial S \partial D}(S'_n, D'_n)\mathbf{d}D_n\mathbf{d}S_n + \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial D^2}(S'_n, D'_n)(\mathbf{d}D_n)^2. \quad (35)\end{aligned}$$

Because Investor plays the strategy given by (10) and (11), the first two terms on the right-hand side of (35) are the capital increment $\mathbf{d}K_n$. So by summing over n and recalling that $\overline{U}(S_0, D_0) = K_0$, one obtains

$$\begin{aligned}&\left|\overline{U}(S_N, D_N) - K_N\right| \\ &\le \frac{1}{2}\sup\left|S^3\frac{\partial^3 \overline{U}}{\partial S^3}\right|\mathbf{var}_S^{\mathrm{rel}}(3) + \frac{1}{2}\sup\left|S^2\frac{\partial^3 \overline{U}}{\partial S^2 \partial D}\right|\sum_n |\mathbf{d}D_n|\left|\frac{\mathbf{d}S_n}{S_n}\right|^2 \\ &+ \sup\left|S\frac{\partial^2 \overline{U}}{\partial S \partial D}\right|\sum_n |\mathbf{d}D_n|\left|\frac{\mathbf{d}S_n}{S_n}\right| + \frac{1}{2}\sup\left|\frac{\partial^2 \overline{U}}{\partial D^2}\right|\mathbf{var}_D(2), \quad (36)\end{aligned}$$

with all suprema over the convex hull of $\{(S_n, D_n) \mid 0 \le n \le N\}$.

The right-hand side of (36), which we want to bound from above, is composed of suprema and sums. The suprema involve partial derivatives; we can bound them using the norms c_2, c_3, and c_4 together with the differential equation (7). The sums are values of $\mathbf{var}_S^{\mathrm{rel}}$ and $\mathbf{var}_D$ or can be related to these variations by classical inequalities; we can bound them using the constraint (3) on Market's moves.

To bound the suprema from above, first use Leibniz's differentiation rule to obtain

$$\begin{aligned}\left|S^n\frac{\partial^n \overline{U}}{\partial S^n}\right| &= \left|S^n\int_{\mathbb{R}} U^{(n)}(Se^z)e^{nz}\ \mathcal{N}_{-D/2,\,D}(dz)\right| \\ &= \left|\int_{\mathbb{R}}(Se^z)^n U^{(n)}(Se^z)\ \mathcal{N}_{-D/2,\,D}(dz)\right| \le \left|\int_{\mathbb{R}} c_n\ \mathcal{N}_{-D/2,\,D}(dz)\right| = c_n. \quad (37)\end{aligned}$$

Then use (7) to express the partial derivatives in (36) as partial derivatives with respect to S alone:

$$\frac{\partial^3 \overline{U}}{\partial S^2 \partial D} = \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(S^2\frac{\partial^2 \overline{U}}{\partial S^2}\right) = \frac{\partial^2 \overline{U}}{\partial S^2} + 2S\frac{\partial^3 \overline{U}}{\partial S^3} + \frac{1}{2}S^2\frac{\partial^4 \overline{U}}{\partial S^4}, \quad (38)$$

$$\frac{\partial^2 \overline{U}}{\partial S \partial D} = \frac{1}{2}\frac{\partial}{\partial S}\left(S^2\frac{\partial^2 \overline{U}}{\partial S^2}\right) = S\frac{\partial^2 \overline{U}}{\partial S^2} + \frac{1}{2}S^2\frac{\partial^3 \overline{U}}{\partial S^3}, \quad (39)$$

$$\frac{\partial^2\overline{U}}{\partial D^2} = \frac{1}{2}\frac{\partial}{\partial D}\left(S^2\frac{\partial^2\overline{U}}{\partial S^2}\right) = \frac{1}{2}S^2\frac{\partial^3\overline{U}}{\partial D\partial S^2} = \frac{1}{2}S^2\frac{\partial^2\overline{U}}{\partial S^2}+S^3\frac{\partial^3\overline{U}}{\partial S^3}+\frac{1}{4}S^4\frac{\partial^4\overline{U}}{\partial S^4}; \quad (40)$$

the last step in (40) uses (38). It follows from (37)–(40) that

$$\left|S^3\frac{\partial^3\overline{U}}{\partial S^3}\right| \le c_3, \quad (41)$$

$$\left|S^2\frac{\partial^3\overline{U}}{\partial S^2\partial D}\right| \le c_2 + 2c_3 + \frac{1}{2}c_4, \quad (42)$$

$$\left|S\frac{\partial^2\overline{U}}{\partial S\partial D}\right| \le c_2 + \frac{1}{2}c_3, \quad (43)$$

$$\left|\frac{\partial^2\overline{U}}{\partial D^2}\right| \le \frac{1}{2}c_2 + c_3 + \frac{1}{4}c_4. \quad (44)$$

Now consider the variations. Set $p := 2+\gamma$ and $q := 2-\gamma$, and note that $p/(p-2) \ge q$. Using (23) and the fact that $p \le 3$, one obtains

$$\sum_n \left|\frac{\mathbf{d}S_n}{S_n}\right|^3 \le \left(\sum_n \left|\frac{\mathbf{d}S_n}{S_n}\right|^p\right)^{3/p} = (\mathbf{var}_S^{\mathrm{rel}}(p))^{3/p} \le \delta^{3/p} \le \delta. \quad (45)$$

Using first Hölder's inequality with $2/p + (p-2)/p = 1$ and then (22) with $p/(p-2) \ge q$, one obtains

$$\begin{aligned}\sum_n |\mathbf{d}D_n| \left|\frac{\mathbf{d}S_n}{S_n}\right|^2 &\le \left(\sum_n \left|\frac{\mathbf{d}S_n}{S_n}\right|^p\right)^{2/p} \left(\sum_n |\mathbf{d}D_n|^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}} \\ &\le \left(\sum_n \left|\frac{\mathbf{d}S_n}{S_n}\right|^p\right)^{2/p} \left(\sum_n |\mathbf{d}D_n|^q\right)^{\frac{p}{(p-2)q}\frac{p-2}{p}} \\ &= (\mathbf{var}_S^{\mathrm{rel}}(p))^{2/p}(\mathbf{var}_D(q))^{1/q} \le \delta^{2/p+1/q} \le \delta^{1+1/p} \le \delta. \quad (46)\end{aligned}$$

Using (24), one obtains

$$\begin{aligned}\sum_n |\mathbf{d}D_n| \left|\frac{\mathbf{d}S_n}{S_n}\right| &\le \left(\sum_n \left|\frac{\mathbf{d}S_n}{S_n}\right|^p\right)^{1/p} \left(\sum_n |\mathbf{d}D_n|^q\right)^{1/q} \\ &= (\mathbf{var}_S^{\mathrm{rel}}(p))^{1/p}(\mathbf{var}_D(q))^{1/q} \le \delta^{1/p}\delta^{1/q} \le \delta. \quad (47)\end{aligned}$$

Finally, using (23), one obtains.

$$\sum_n |\mathbf{d}D_n|^2 \le \left(\sum_n |\mathbf{d}D_n|^q\right)^{2/q} = (\mathbf{var}_D(q))^{2/q} \le \delta^{2/q} \le \delta. \quad (48)$$

Combining (36) with (41)–(44) and (45)–(48), one obtains the bound (32). This completes the proof of the lemma. ∎

To complete the proof of Proposition 1, one must show that U can be approximated by a smooth function V that has norms c_2, c_3, and c_4 small enough that $40c\delta^{1/4}$ bounds the sum of (32) and the additional error caused by the difference between U and V.

Let us define V by smoothing U on a logarithmic scale for the security price S. Formally, this involves three steps:

- Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := U(e^x)$.
- Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := \int f(x+z)\,\mathcal{N}_{0,\sigma^2}(dz)$ for some $\sigma^2 > 0$.
- Define $V : (0,\infty) \to: \mathbb{R}$ by $V(S) := g(\ln S)$.

Because U is log-Lipschitzian with coefficient c, f is Lipschitzian with coefficient c.

First let us check that $U(S)$ is close to $V(S)$ or, equivalently, that f is close to g. Using f's being Lipschitzian, we obtain

$$\begin{aligned}|V(S) - U(S)| = |g(\ln S) - f(\ln S)| &= \left|\int_{\mathbb{R}} f(\ln S + z) - f(\ln S)\ \mathcal{N}_{0,\sigma^2}(dz)\right| \\ &\le \int_{\mathbb{R}} |f(\ln S + z) - f(\ln S)|\ \mathcal{N}_{0,\sigma^2}(dz) \le c\int_{\mathbb{R}} |z|\ \mathcal{N}_{0,\sigma^2}(dz) \\ &= c\sigma \int_{\mathbb{R}} |z|\ \mathcal{N}_{0,1}(dz) = \sqrt{2/\pi}c\sigma; \quad (49)\end{aligned}$$

the last equality can be obtained using (25) and (26) to evaluate the integral. We also obtain

$$\begin{aligned}|\overline{V}(S,D) - \overline{U}(S,D)| &= \left|\int_{\mathbb{R}} V(Se^z) - U(Se^z)\ \mathcal{N}_{-D/2,\,D}(dz)\right| \\ &\le \int_{\mathbb{R}} |V(Se^z) - U(Se^z)|\ \mathcal{N}_{-D/2,\,D}(dz) \le \sqrt{2/\pi}\,c\sigma; \quad (50)\end{aligned}$$

the last inequality follows from (49).

Now we find upper bounds for the derivatives of g. Assuming, without loss of generality, that $f(\ln S) = 0$ for the particular value of S we are considering, and using Lemma 1, we obtain for $n = 1, 2, \ldots$:

$$\begin{aligned}|g^{(n)}(\ln S)| &\le \frac{1}{\sqrt{2\pi}\sigma^{n+1}} \int_{\mathbb{R}} e^{-(x-\ln S)^2/(2\sigma^2)} \left|H_n\left(\frac{x - \ln S}{\sigma}\right)\right| c|x - \ln S|\,dx \\ &= \frac{c}{\sqrt{2\pi}\sigma^{n-1}} \int_{\mathbb{R}} e^{-y^2/2}|H_n(y)||y|\,dy, \quad (51)\end{aligned}$$

Since the last expression does not involve S, we see that the norm $\|g^{(n)}\| := \sup_{x\in(0,\infty)}$ is bounded by

$$\|g^{(n)}\| \le \frac{c}{\sqrt{2\pi}\sigma^{n-1}} \int_{\mathbb{R}} e^{-y^2/2}|H_n(y)||y|\,dy. \quad (52)$$

Evaluating the integral using (25) and (26), we find

$$\|g^{(1)}\| \le \frac{c}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-y^2/2} |y|^2 \, dy = c,$$

$$\begin{aligned}\|g^{(2)}\| &\le \frac{c}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} e^{-y^2/2} |y^2 - 1| |y| \, dy \\ &\le \frac{2c}{\sqrt{2\pi}\sigma} \int_0^\infty e^{-y^2/2} (y^3 + y) \, dy = \frac{2c}{\sqrt{2\pi}\sigma}(2+1) = \frac{3\sqrt{2}c}{\sqrt{\pi}\sigma},\end{aligned}$$

$$\begin{aligned}\|g^{(3)}\| &\le \frac{c}{\sqrt{2\pi}\sigma^2} \int_{\mathbb{R}} e^{-y^2/2} |y^3 - 3y| |y| \, dy \\ &\le \frac{c}{\sqrt{2\pi}\sigma^2} \int_{\mathbb{R}} e^{-y^2/2} (y^4 + 3y^2) \, dy = \frac{6c}{\sigma^2},\end{aligned}$$

and

$$\begin{aligned}\|g^{(4)}\| &\le \frac{c}{\sqrt{2\pi}\sigma^3} \int_{\mathbb{R}} e^{-y^2/2} |y^4 - 6y^2 + 3| |y| \, dy \\ &\le \frac{2c}{\sqrt{2\pi}\sigma^3} \int_0^\infty e^{-y^2/2} (y^5 + 6y^3 + 3y) \, dy = \frac{2c}{\sqrt{2\pi}\sigma^3}(8 + 12 + 3) = \frac{23\sqrt{2}c}{\sqrt{\pi}\sigma^3};\end{aligned}$$

Using these bounds on $\|g^{(n)}\|$, we obtain for V's norms (in the sense of (31)):

$$\left|S^2 V^{(2)}\right| = \left|S^2 \frac{d^2}{dS^2} g(\ln S)\right| = |g''(\ln S) - g'(\ln S)| \le \frac{3\sqrt{2}c}{\sqrt{\pi}\sigma} + c,$$

$$\begin{aligned}\left|S^3 V^{(3)}\right| = \left|S^3 \frac{d^3}{dS^3} g(\ln S)\right| &= \left|g^{(3)}(\ln S) - 3g''(\ln S) + 2g'(\ln S)\right| \\ &\le \frac{6c}{\sigma^2} + 3\frac{3\sqrt{2}c}{\sqrt{\pi}\sigma} + 2c,\end{aligned}$$

$$\begin{aligned}\left|S^4 V^{(4)}\right| = \left|S^4 \frac{d^4}{dS^4} g(\ln S)\right| &= \left|g^{(4)}(\ln S) - 6g^{(3)}(\ln S) + 11g''(\ln S) - 6g'(\ln S)\right| \\ &\le \frac{23\sqrt{2}c}{\sqrt{\pi}\sigma^3} + 6\frac{6c}{\sigma^2} + 11\frac{3\sqrt{2}c}{\sqrt{\pi}\sigma} + 6c.\end{aligned}$$

Lemma 2 tells us that if Investor plays the strategy computed from V starting with the capital $\overline{V}(S_0, D_0)$, he will hit the target payoff $V(S_N)$ with error at most

$$\delta\left(0.375\frac{23\sqrt{2}c}{\sqrt{\pi}\sigma^3} + 4.75\frac{6c}{\sigma^2} + 13.375\frac{3\sqrt{2}c}{\sqrt{\pi}\sigma} + 9c\right).$$

Equation (49) tells us that $V(S_N)$ will differ from $U(S_N)$ by at most $\sqrt{2/\pi}c\sigma$, and Equation (50) tells us that $\overline{V}(S_0, D_0)$ will differ from $\overline{U}(S_0, D_0)$ by at most $\sqrt{2/\pi}c\sigma$. So the strategy computed from V misses by at most

$$2\sqrt{2/\pi}c\sigma + \delta\left(0.375\frac{23\sqrt{2}c}{\sqrt{\pi}\sigma^3} + 4.75\frac{6c}{\sigma^2} + 13.375\frac{3\sqrt{2}c}{\sqrt{\pi}\sigma} + 9c\right)$$

when it starts with $\overline{U}(S_0, D_0)$ and tries to hit $U(S_N)$.

This expression has the form

$$A\sigma + B\sigma^{-3} + C\sigma^{-2} + D\sigma^{-1} + E. \tag{53}$$

Let us set

$$\sigma := \left(\frac{A}{3B}\right)^{-1/4}$$

(this is the value of σ minimizing $A\sigma + B\sigma^{-3}$); the expression (53) then becomes

$$\begin{aligned}
&(3^{1/4} + 3^{-3/4})A^{3/4}B^{1/4} + 3^{-1/2}A^{1/2}B^{-1/2}C + 3^{-1/4}A^{1/4}B^{-1/4}D + E \\
&= (3^{1/4} + 3^{-3/4})\left(2\sqrt{2/\pi}c\right)^{3/4}\left(0.375\frac{23\sqrt{2}}{\sqrt{\pi}}c\delta\right)^{1/4} \\
&+3^{-1/2}\left(2\sqrt{2/\pi}c\right)^{1/2}\left(0.375\frac{23\sqrt{2}}{\sqrt{\pi}}c\delta\right)^{-1/2} 4.75\times 6c\delta \\
&+3^{-1/4}\left(2\sqrt{2/\pi}c\right)^{1/4}\left(0.375\frac{23\sqrt{2}}{\sqrt{\pi}}c\delta\right)^{-1/4} 13.375\frac{3\sqrt{2}}{\sqrt{\pi}}c\delta + 9c\delta \\
&\le \delta^{1/4}c\Big((3^{1/4} + 3^{-3/4})2^{5/4}\pi^{-1/2}0.375^{1/4}23^{1/4} \\
&+2^{3/2}3^{1/2}0.375^{-1/2}23^{-1/2}4.75 + 2^{3/4}3^{3/4}\pi^{-1/2}0.375^{-1/4}23^{-1/4}13.375 + 9\Big) \\
&\le 37.84c\delta^{1/4}.
\end{aligned}$$

This completes the proof.

A.3 Using the proof

For relatively smooth payoff functions, the bound provided by Lemma 2, $\delta\left(1.75c_2 + 2.5c_3 + 0.375c_4\right)$, is much tighter than the general bound $40c\delta^{1/4}$. It may be possible to improve the latter by improving the approximations in the proof, but it seems likely that any bound that applies to so wide a class of payoff functions will be too loose to be useful.

As a practical matter, one does not want to hedge all possible options; one wants to hedge particular options. So instead of seeking a general bound, one should use the lemma to find a tight bound for each particular option. This means finding a smooth function that comes close to the option's payoff function but has a small value for $\delta\,(1.75c_2 + 2.5c_3 + 0.375c_4)$. The error for the particular option can then be bounded by this bound plus twice the maximum distance between the two functions.

This should work even for an option, such as the European call, that has an infinite value for the log-Lipschitzian c.

B Clarifications

The following notes attempt to clarify the relation between the ideas of this article and other ideas in the literature.

B.1 Why do security prices scale as $\sqrt{\mathbf{d}t}$?

The assumption of $\sqrt{\mathbf{d}t}$ scaling for price fluctuations can justified by an arbitrage argument. Were the fluctuations to scale in any other way with time, an arbitrageur could make money with a simple strategy—a momentum strategy if the average magnitude of fluctuations decreases slower than $\sqrt{\mathbf{d}t}$ does as $\mathbf{d}t$ gets smaller, a contrarian strategy if it decreases faster. In recent years this has been explained by several authors under probabilistic assumptions. It is known for example, that arbitrage is possible if a price process follows fractional Brownian motion with scaling different from $\sqrt{\mathbf{d}t}$ [6].

It might be thought that this arbitrage argument depends in some way on the assumption that the prices are random or even on the assumption that the prices follow a particular probability model such as fractional Brownian motion. Recent work shows, however, that the arbitrage argument can be formulated in discrete non-probabilistic terms, in the spirit of this article. So we can say that approximate $\sqrt{\mathbf{d}t}$ scaling is required in order to avoid arbitrage in discrete time [15].

The argument for $\sqrt{\mathbf{d}t}$ scaling does not apply to $\mathcal{D}$, because of its constant stream of dividends. Indeed, $\mathcal{D}$'s price changes should be dominated by its decline in value as it pays out its dividends. Since these payouts themselves scale as $\mathbf{d}t$, we can expect $\mathcal{D}$'s price changes to scale as $\mathbf{d}t$. This is only a plausibility argument, but it gains empirical support from data presented on pp. 254–275 of [12]. It would be interesting to see if an arbitrage argument could be made for the scaling of $\mathcal{D}$'s price changes.

B.2 What is the role of probability theory?

The established theory of option pricing depends only superficially on the assumption that security prices are stochastic. In order to demonstrate mathematically that an option can be replicated dynamically by trading in the underlying security, one usually assumes that the path followed by the price of the security is governed by a specific probability measure. But the demonstration uses little of the information in the probability measure. It uses only the fact that a particular set of paths (all with $\sqrt{\mathbf{d}t}$ scaling) is assigned probability one. How the probabilities are distributed over the paths within this set does not matter.

Some effort has been made to clarify this point by formulating the idea of $\sqrt{\mathbf{d}t}$ scaling independent of probability theory. Bick and Willinger [2], for example, formalize the idea of $\sqrt{\mathbf{d}t}$ scaling in continuous time non-probabilistically, using non-standard analysis, and show that an option on $\mathcal{S}$ can be replicated at a cost equal to its Black-Scholes price if $\mathcal{S}$'s returns scale exactly as $\sqrt{\mathbf{d}t}$ and $\mathbf{var}_S(2)$ is constant. But the appeal to non-standard analysis makes this continuous-time result seem as remote as the probabilistic results. What relation does it have to discrete-time financial reality?

It appears that continuous-time theory, whether probabilistic or not, is misleading. The limits of arbitrage in discrete time are too broad for delta-hedging alone to achieve what it can do theoretically in continuous time. Additional help is needed from supply and demand. We need the market to help by pricing additional instruments, either calls and puts as it does now, or a battery of variance instruments such as those proposed in this article.

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