A Game-Theoretic Explanation of the $\sqrt{dt}$ Effect

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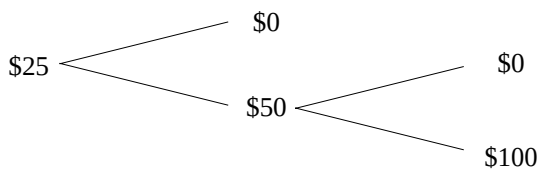

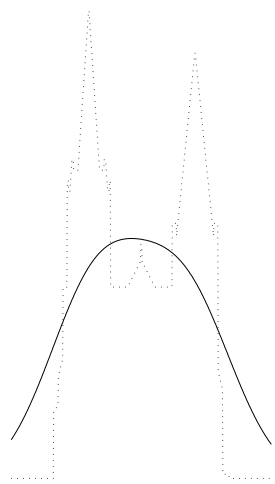

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Abstract

The returns from a traded security typically have the order of magnitude $\sqrt{dt}$. This can be explained by the absence of riskless opportunities for making money: any different order of magnitude would permit such opportunities. The explanation has been made rigorous in the standard measure-theoretic framework for continuous-time probability, but this requires the assumption that the security's price process follows a particular stochastic process (usually fractional Brownian motion). This note makes the explanation rigorous without any stochastic assumption, using our purely game-theoretic framework for finance [7]. This is a preliminary draft.

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1 Introduction

The returns from a traded security typically have the order of magnitude $\sqrt{dt}$. This can be explained by the absence of riskless opportunities for making money: any different order of magnitude would permit such opportunities. It is easy to elaborate this explanation at an intuitive level, where one speaks freely about infinitesmals, because very simple strategies are sufficient to take advantage of returns with an order of magnitude different from $\sqrt{dt}$. In this note, we show how to translate the intuitive explanation into the purely game-theoretic framework for probability and finance developed in our book, *Probability and Finance: It's Only a Game!* [7].

Our game-theoretic result should be compared with analogous measure-theoretic results obtained by Rogers [6], Delbaen and Schachermayer [4], and Cheridito [1, 2]. We rely on the same intuition as these authors,[1] but our result, in contrast to their results, does not rely on stochastic assumptions. In order to use the measure-theoretic framework, one must assume that the price of the security follows some stochastic process, and for technical reasons one usually considers some very narrow class of processes, such as fractional or exponential fractional Brownian motion.

As we explain in Appendix 11.5 of our book, our continuous-time framework is an infinitary game. Intuitively, it is a game with an infinite number of rounds N, each taking an infinitesmal period of time dt, during which the price S of the security changes by dS. Mathematically, it is an ultraproduct of an infinite sequence of finitary games, which divide a finite time period $[0, T]$ into shorter and shorter trading periods. (The same notion of ultraproduct is used in nonstandard analysis to construct the hyperreals from countably many copies of the reals.) In this infinitary game, we define the *p-variation* of a price process S, for any real number $p > 0$, as an infinite sum:

$$\mathbf{var}_S(p) := \sum_{i=0}^{N-1} |dS_n|^p.$$

(We use the notation of [7]: $\Delta f_n := f_n - f_{n-1}$, while $df_n := f_{n+1} - f_n$.) There will be some number, which we designate $\operatorname{vex} S$, such that $\mathbf{var}_S(p)$ is infinite for $p > \operatorname{vex} S$ and infinitesmal for $p < \operatorname{vex} S$. We call $\operatorname{vex} S$ the *volatility exponent* for S. The inverse of $\operatorname{vex} S$ is the *Hölder exponent* or the *Hurst exponent*. The statement that the increments dS have the order of magnitude $\sqrt{dt}$ becomes, in this framework, the statement that $\operatorname{vex} = 2$, or that the Hölder/Hurst exponent is $1/2$.

We prove our infinitary result in §2, using relatively informal language that can be understood rigorously by those familiar with the idea of an ultraproduct. For those not so familiar with this idea, we provide some additional explanation in an appendix.

[1] In fact, this note results from Delbaen's calling our attention to Cheridito's work in his review of our book [3]. For our response to Delbaen's review, see www.probabilityandfinance.com.

In §3 and §4. we give messier forms of our result in a realistic, discrete-time setting. We report the results of (extremely) preliminary empirical studies in §5. In a final section, §7, we remove the assumption of zero interest rate.

This is a preliminary draft. In later drafts, we intend to give more explanation of the infinitary game and to further simplify the infinitary picture by using the alternative definition of game-theoretic upper probability zero discussed in Frequently Asked Question #1 at www.probabilityandfinance.com.

2 Nonstandard results

The basic framework is that of Chapter 11: the interval T is split into an infinitely large number N of subintervals etc.

THE MARKET PROTOCOL
Players: Investor, Market
Protocol:
$\mathcal{I}_0 := 1$.
Market announces $S_0 \in \mathbb{R}$.
FOR $n = 1, 2, \ldots, N$:
Investor announces $M_n \in \mathbb{R}$.
Market announces $S_n \in \mathbb{R}$.
$\mathcal{I}_n := \mathcal{I}_{n-1} + M_n \Delta S_n$.
Additional Constraint on Market: Market must ensure that S is continuous.

The definition of zero game-theoretic probability is given on pp. 340–341 of [7]: an event E has zero game-theoretic probability if for any K there exists a strategy that, when started with 1, does not risk bankruptcy and finishes with capital at least K when E happens.

We start with a result showing that too low volatility gives opportunities for making money.

Theorem 1 *For any $\delta > 0$, the event*

$$\operatorname{vex} S < 2 \ \& \ \sup_t |S(t) - S(0)| > \delta$$

has game-theoretic probability zero.

The condition $\operatorname{vex} S < 2$ means that S is less volatile than the Brownian motion and $\sup_t |S(t) - S(0)| > \delta$ means that S should not be almost constant.

Proof This proof is a simple modification of Example 3 in [8], p. 658, and a proof in [1, 2]. It is given in the usual style of [7]; in the appendix we will provide additional details.

Assume, without loss of generality, that $S(0) = 0$ (if this is not true, replace $S(t)$ by $S(t) - S(0)$. Consider the strategy $M_n := 2CS_n$, where C is a large positive constant. With our usual notation $df_n := f_{n+1} - f_n$, we have

$$d\mathcal{I}_n = 2CS_n dS_n = C\left(d(S_n^2) - (dS_n)^2\right)$$

and, therefore,

$$\mathcal{I}_n - \mathcal{I}_0 = CS_n^2 - C\sum_{i=0}^{n-1}(dS_i)^2 \approx CS_n^2. \tag{1}$$

If this strategy starts with 1, the capital at each step n will be nonnegative. Stopping playing at the first step when $|S_n| > \delta$, we make sure that $\mathcal{I}_N \geq C\delta^2$, which can be made arbitrarily large by taking a large C. ∎

As the proof shows, the condition $\operatorname{vex} S < 2$ of the theorem can be replaced by the weaker $\mathbf{var}_S(2) = 0$.

Now we complement Theorem 1 with a result dealing with too high volatility.

Theorem 2 *For any $D > 0$, the event*

$$\operatorname{vex} S > 2 \;\&\; \sup_t |S(t) - S(0)| < D \tag{2}$$

has game-theoretic probability zero.

Proof This proof is a simple modification of a proof in [1, 2]. We again assume $S(0) = 0$.

Consider the strategy $M_n := -2D^{-2}S_n$. Now we have

$$d\mathcal{I}_n = -2D^{-2}S_n dS_n = D^{-2}\left((dS_n)^2 - d(S_n^2)\right)$$

and, therefore,

$$\mathcal{I}_n - \mathcal{I}_0 = D^{-2}\sum_{i=0}^{n-1}(dS_i)^2 - D^{-2}S_n^2 \geq D^{-2}\sum_{i=0}^{n-1}(dS_i)^2 - 1$$

before $|S_n|$ reaches D. If this strategy starts with 1 and stops playing as soon as S_n reaches D, the capital at each step n will be nonnegative and, if event (2) occurs, $\mathcal{I}_N \geq D^{-2}\,\mathbf{var}_S(2)$ will be infinitely large. ∎

As before, the condition $\operatorname{vex} S > 2$ can be replaced by $\mathbf{var}_S(2) = \infty$.

If S is a stock price, it cannot become negative, which allows us to strengthen the conclusion of Theorem 2.

Corollary 1 *The event* $\operatorname{vex} S > 2$ *has game-theoretic probability zero.*

Proof Let K be the constant from the definition of zero game-theoretic probability (p. 2). The required strategy is the 50/50 mixture of the following 2 strategies: the strategy of Theorem 2 corresponding to $D := 2K$ and the buy-and-hold strategy that recommends buying 1 share of S at the outset. If $\sup_t |S(t) - S(0)| < 2K$, the first strategy will make Investor rich; otherwise, the second will. ∎

3 Absolute finitary result

The protocol for this section is:

THE ABSOLUTE MARKET PROTOCOL
Players: Investor, Market
Protocol:
$\mathcal{I}_0 := 1$.
Market announces $S_0 \in \mathbb{R}$.
FOR $n = 1, 2, \ldots, N$:
Investor announces $M_n \in \mathbb{R}$.
Market announces $S_n \in \mathbb{R}$.
$\mathcal{I}_n := \mathcal{I}_{n-1} + M_n \Delta S_n$.

Now N is a usual positive integer number and there are no *a priori* constraints on Market. The following is the "absolute" finitary version of Theorem 1.

Theorem 3 *Let ϵ and δ be two positive numbers. If Market is required to satisfy*

$$\sum_{i=1}^{N} (\Delta S_i)^2 \le \epsilon,$$

the game-theoretic probability of the event

$$\max_{n=1,\ldots,N} |S_n - S_0| \ge \delta \tag{3}$$

is at most ϵ/δ^2.

Proof Assume, without loss of generality, that $S_0 = 0$ (replace S_n by $S_n - S_0$ if not). Take the same strategy $M_n := 2CS_n$, but now $C = 1/\epsilon$. From (1) we obtain

$$\mathcal{I}_n - \mathcal{I}_0 = \frac{1}{\epsilon} S_n^2 - \frac{1}{\epsilon} \sum_{i=0}^{n-1} (dS_i)^2 \ge \frac{1}{\epsilon} S_n^2 - 1,$$

i.e., if the strategy starts with 1,

$$\mathcal{I}_n \ge \frac{1}{\epsilon} S_n^2.$$

This shows that $\mathcal{I}_n$ is never negative; stopping at the step n where $|S_n| \ge \delta$, we make sure that $\mathcal{I}_N \ge \delta^2/\epsilon$ when (3) happens. ∎

4 Relative finitary result

Now we change our protocol to:

The Relative Market Protocol
Players: Investor, Market
Protocol:
$\mathcal{I}_0 := 1$.
Market announces $S_0 > 0$.
FOR $n = 1, 2, \ldots, N$:
Investor announces $M_n \in \mathbb{R}$.
Market announces $S_n > 0$.
$\mathcal{I}_n := \mathcal{I}_{n-1} + M_n \Delta S_n$.

As in the previous section, N is a standard positive integer number. Define a nonnegative function β by

$$\frac{1}{2}\beta(x) = x - \ln(1+x);$$

so for small $|x|$, $\beta(x)$ behaves as x^2. The following result is the "relative" finitary version of Theorem 1; it uses the versions

$$\sum_{i=0}^{N-1} (d \ln S_i)^2$$

and

$$\sum_{i=0}^{N-1} \beta\left(\frac{dS_i}{S_i}\right)$$

of the 2-variation

$$\sum_{i=0}^{N-1} \left(\frac{dS_i}{S_i}\right)^2$$

of S.

Theorem 4 *Let ϵ, δ and γ be three positive numbers. If Market is required to satisfy*

$$\sum_{i=0}^{N-1} (d \ln S_i)^2 \le \epsilon, \quad \sum_{i=0}^{N-1} \beta\left(\frac{dS_i}{S_i}\right) \le \epsilon$$

and

$$\min_n \ln S_n \ge -\gamma,$$

the game-theoretic probability of the event

$$\max_{n=1,\ldots,N} |\ln(S_n/S_0)| \ge \delta \tag{4}$$

is at most $(1+\gamma)\epsilon/\delta^2$.

Proof Assume, without loss of generality, that $S_0 = 1$ (replace S_n by S_n/S_0 if not). Since the proof is now slightly more complicated than that in the previous section, we first outline its idea. Roughly speaking, our goal will be to maintain $\mathcal{I}_n$ close to $(\ln S_n)^2$ (in the previous sections it was to maintain $\mathcal{I}_n$ close to $(S_n)^2$). To find a strategy that will achieve this, we notice that

$$\begin{aligned} d\left(\ln^2 S_n\right) &= (2\ln S_n)(d\ln S_n) + (d\ln S_n)^2 && (5)\\ &= (2\ln S_n)\ln\left(1+\frac{dS_n}{S_n}\right) + (d\ln S_n)^2 && (6)\\ &= (2\ln S_n)\frac{dS_n}{S_n} - (\ln S_n)\beta\left(\frac{dS_n}{S_n}\right) + (d\ln S_n)^2. && (7) \end{aligned}$$

We can see that a suitable strategy is

$$M_n := 2C\frac{\ln S_n}{S_n}$$

for some C (chosen so that to make sure that the capital process is nonnegative; eventually we will take $C = 1/((1+\gamma)\epsilon)$). Expressing $2(\ln S_n)(dS_n)/S_n$ from the equality between the extreme terms of the chain (5)–(7), we obtain for the strategy M_n:

$$d\mathcal{I}_n = 2C\frac{\ln S_n}{S_n}dS_n = Cd\ln^2 S_n + C\ln S_n\beta\left(\frac{dS_n}{S_n}\right) - C(d\ln S_n)^2;$$

therefore,

$$\begin{aligned} \mathcal{I}_n - \mathcal{I}_0 &= C\ln^2 S_n + C\sum_{i=0}^{n-1}(\ln S_i)\beta\left(\frac{dS_i}{S_i}\right) - C\sum_{i=0}^{n-1}(d\ln S_i)^2 && (8)\\ &\geq C\ln^2 S_n - C\gamma\epsilon - C\epsilon. && (9) \end{aligned}$$

Starting from $\mathcal{I}_0 = 1$, it is safe to take $C := 1/((1+\gamma)\epsilon)$ (this removes possibility of bankruptcy), in which case (8)–(9) becomes

$$\mathcal{I}_n \geq \frac{1}{(1+\gamma)\epsilon}\ln^2 S_n.$$

Stopping at the step n with $|\ln S_n| \geq \delta$ ensures $\mathcal{I}_N \geq \delta^2/((1+\gamma)\epsilon)$ when (4) happens. ∎

5 Empirical studies

The empirical studies reported in this section are closely connected to the so called $\mathcal{R}/\mathcal{S}$-analysis (see [8], §4a). The results we report here assume zero interest rate, and so are of limited interest; this will be corrected in the future versions.

First we consider the absolute setting, although the usual definitions as given in [8] are "relative". Denote

$$\mathcal{R}_N^{\text{abs}} := \max_{i=1,\dots,N} |S_n - S_0|, \quad \left(\mathcal{S}_N^{\text{abs}}\right)^2 := \frac{1}{N}\sum_{i=1}^{N}(\Delta S_i)^2.$$

Suppose that we believe, for some reason, that we are going to have $\mathcal{S}_N^{\text{abs}} \le \sigma$ and $\mathcal{R}_N^{\text{abs}} \ge \delta$; therefore, δ plays the same role as in Theorem 3 and σ plays the role of $\sqrt{\epsilon/N}$. So from Theorem 3 we obtain that we will be able to multiply our capital $\delta^2/\epsilon = (\delta/\sigma)^2/N$-fold. For another variant of the definitions of $\mathcal{R}_N$ and $\mathcal{S}_N$ (as given in [8], (14) on p. 371; see below) one usually has

$$\frac{\mathcal{R}_N}{\mathcal{S}_N} \sim cN^{\mathbb{H}}$$

with $\mathbb{H}$ considerably larger than 1/2. If our guesses δ and σ are not too far off, we can hope to increase our initial capital by a factor of order $N^{2\mathbb{H}-1}$.

In the "relative" setup, define

$$\mathcal{R}_N^{\text{rel}} := \max_{n=1,\dots,N} \left|\ln \frac{S_n}{S_0}\right|, \quad \left(\mathcal{S}_N^{\text{abs}}\right)^2 := \frac{1}{N}\left(\sum_{i=0}^{N-1} \beta\left(\frac{dS_i}{S_i}\right) \bigvee \sum_{i=0}^{N-1} (d\ln S_i)^2\right).$$

If we believe that we are going to have $\mathcal{S}_N^{\text{rel}} \le \sigma$ and $\mathcal{R}_N^{\text{rel}} \ge \delta$, we obtain from Theorem 4 that we will be able to multiply our capital by a factor of

$$\frac{\delta^2}{(1+\gamma)\epsilon} = \frac{\delta^2}{(1+\gamma)\sigma^2 N}$$

(where $\epsilon := \sigma^2 N$); if one has

$$\frac{\mathcal{R}_N}{\mathcal{S}_N} \sim cN^{\mathbb{H}}$$

and our guesses δ and σ are not too far off, we can again hope to increase our initial capital by a factor of order

$$N^{2\mathbb{H}-1}. \tag{10}$$

Some experimental results are given in [8] (§4.4), but we cannot use them directly, since the standard definitions of $\mathcal{R}/\mathcal{S}$ analysis are different from ours (the main difference being that the standard definition are centered). Those results, however, suggest that typically $\mathbb{H} > 0.5$, which was why we concentrated on this case in our discrete-time analysis and empirical studies.

In our experiments we considered, instead of $\mathcal{R}_N^{\text{abs}}$ and $\mathcal{R}_N^{\text{rel}}$, $|S_N - S_0|$ and $|\ln(S_N/S_0)|$, respectively, the rationale being that security prices typically increase. This free us from the need to guess the value of δ in advance. Our results are summarized in Tables 1 and 2.

In Table 1 we list the 19 securities for which we conducted experiments. The number N is the number of trading periods (days, month, or years).

Security and frequency	Code	Time Period	N
Microsoft stock daily	msft d	13/03/1986–21/09/2001	3672
IBM stock daily	ibm d	02/01/1962–21/09/2000	9749
S&P500 daily	spc d	04/01/1960–21/09/2000	10,254
Microsoft stock monthly	msft m	March 1986–June 2001	184
IBM stock monthly	ibm m	January 1962–June 2001	474
General Electric stock monthly	ge m	January 1962–June 2001	474
Boeing stock monthly	ba m	January 1970–June 2001	378
Du Pont (E.I.) de Nemours stock monthly	dd m	January 1970–June 2001	378
Consolidated Edison stock monthly	ed m	January 1970–June 2001	378
Eastman Kodak stock monthly	ek m	January 1970–June 2001	378
General Motors stock monthly	gm m	January 1970–June 2001	378
Procter and Gamble stock monthly	pg m	January 1970–June 2001	378
Sears/Roebuck stock monthly	s m	January 1970–June 2001	378
AT&T stock monthly	t m	January 1970–June 2001	378
Texaco stock monthly	tx m	January 1970–June 2001	378
US T-bill monthly	us m	January 1871–June 2001	1566
S&P500 Total Returns monthly	sp m	January 1871–June 2001	1566
US T-bill yearly	us a	1871–2000	130
S&P500 Total Returns yearly	sp a	1871–2001	130

Table 1: The 19 securities used in our experiments. Dates are given in the format dd/mm/yyyy.

The numbers given in Table 2 are defined as follows:

$$\text{abs factor} := \frac{(S_N - S_0)^2}{\sum_{i=0}^{N-1}(dS_i)^2}$$

and

$$\text{rel factor} := \frac{\left(\ln \frac{S_N}{S_0}\right)^2}{(1-\min)\left(\sum_{i=0}^{N-1}(d\ln S_i)^2 \ \bigvee\ \sum_{i=0}^{N-1}\beta\left(\frac{dS_i}{S_i}\right)\right)},$$

where

$$\min := \min_n \ln \frac{S_n}{S_0}.$$

To judge the magnitude of abs factor and rel factor we also give the factor by which the value of the security increases (the column "security") and the factor by which the value of an index (S&P500) increases (the column "index") over the same time period.

As we already mentioned, our experiments implicitly assume zero interest rate, but the results they give are roughly of the same order of magnitude as those implied by the table on p. 376 of [8]. Line 1 of that table can be interpreted (ignoring the facts that centering is not the same thing as discounting and that

code	abs factor	rel factor	index	security	min
msft d	1.32	13.8	6.13	330	−0.0736
ibm d	2.62	1.94	20.8	16.0	−0.626
spc d	8.25	10.7	24.2	24.2	−0.138
msft m	0.930	12.6	7.59	382	0
ibm m	2.08	2.22	69.4	15.8	−0.469
ge m	4.57	6.99	69.4	63.4	−0.232
ba m	2.35	3.25	43.5	75.5	−0.620
dd m	1.66	6.72	43.5	33.0	0
ed m	4.21	3.76	43.5	58.0	−1.19
ek m	0.638	0.994	43.5	5.06	−0.441
gm m	1.41	2.24	43.5	13.8	−0.462
pg m	1.20	4.88	43.5	18.6	−0.166
s m	0.0331	0.0223	43.5	1.37	−0.723
t m	0.0293	0.0159	43.5	0.702	−1.13
tx m	4.66	7.97	43.5	36.7	0
us m	3.87	1081	91600	282	0
sp m	6.99	32.9	91,600	91,600	−0.0877
us a	32.2	93	87600	261	0
sp a	6.57	23.9	87,600	87,600	−0.00663

Table 2: Empirical results related to Theorems 3 and 4.

DJIA cannot be reproduced by a trading strategy) as saying that our initial capital can be increased by a factor of roughly

$$12,500^{2\times 0.59-1} \approx 5.46$$

in 12,500 days since 1888.

6 Axiom of continuity for upper probability?

We define upper probability P in terms of strategies, and it may not follow from our definitions that it is continuous in the sense

$$A_1 \subseteq A_2 \subseteq \cdots \Longrightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} P(A_i).$$

We can, however, "regularize" P replacing it by

$$\overline{P}(A) := \inf\left\{\lim_{i\to\infty} P(A_i) \,|\, A_1 \subseteq A_2 \subseteq \cdots \;\&\; A \subseteq \bigcup_{i=1}^{\infty} A_i\right).$$

We do not seriously propose this step at this point, because its only justification seems to be mathematical convenience. The situation is, however,

analogous with Kolmogorov's axiom of continuity at the time of writing [5]; for further discussion, see www.probabilityandfinance.com. If the game-theoretic axiom of continuity does not lead to counter-intuitive or mathematically awkward results, it may be convenient to accept it, but it is too early to make a definitive judgment about its merits yet.

It is clear that the continuity axiom simplifies the above Theorems 1 and 2. We say that an event is *full* if its complement has zero regularized game-theoretic probability, and we write $S \approx \text{const}$ to mean that $\sup_t |S(t) - S(0)|$ is infinitesimal.

Theorem 5 *The event*

$$\operatorname{vex} S = 2 \text{ or } S \approx \text{const}$$

is full.

7 Non-zero interest rate

Our protocols implicitly assume that the interest rate is zero. In this section we remove this restriction. Our protocol now involves not only security S but also another security B (e.g., a bond). Their prices are assumed nonnegative.

The Market Protocol
Players: Investor, Market
Protocol:
$\mathcal{I}_0 := 1$.
Market announces $S_0 > 0$ and $B_0 > 0$.
FOR $n = 1, 2, \dots, N$:
Investor announces $M_n \in \mathbb{R}$ and .
Market announces $S_n \in \mathbb{R}$.
$\mathcal{I}_n := (\mathcal{I}_{n-1} - M_n S_{n-1}) \frac{B_n}{B_{n-1}} + M_n S_n$.
Additional Constraint on Market: Market must ensure that S and B are continuous.

(Cf. the protocol and its analysis on p. 296 of [7]). Intuitively, at step n Investor buys M_n units of S and invests the remaining money in B, which can be a money market account, a bond, or any other security with nonnegative prices. The protocol of §2 corresponds to a constant B_n.

Re-expressing Investor's capital and the price of S in the *numéraire* B_n, we obtain

$$\mathcal{I}_n^\dagger := \mathcal{I}_n / B_n, \quad S_n^\dagger := S_n / B_n.$$

It is easy to see that

$$\mathcal{I}_n^\dagger := \mathcal{I}_{n-1}^\dagger - M_n S_{n-1}^\dagger + M_n S_n^\dagger,$$

which is exactly the expression that we had in §2, only with the daggers added. Therefore, we can restate all results of §2 for the current protocol. For example, Theorem 5 implies:

Theorem 6 *The event*

$$\operatorname{vex}(S/B) = 2 \text{ or } S/B \approx \text{const}$$

is full.

We have an interesting all-or-nothing phenomenon: either two securities are proportional or their ratio behaves stochastically.

Acknowledgements

The theorems in § 2 were inspired by questions raised by Freddy Delbaen in his review of our book in the *Journal of the American Statistical Association*. Our reply to his review is available at www.probabilityandfinance.com.

Appendix: Details of the proof of Theorem 1

Let us show more formally why $\mathcal{I}_n$ is nonnegative and why $\mathcal{I}_N \geq C\delta^2$.

According to the first equality in (1), in every finitary structure we have

$$\mathcal{I}_n - 1 \geq -C \sum_{i=0}^{N-1} (dS_i)^2;$$

since the value on the right-hand side is infinitesimal (and, therefore, smaller than 1 in absolute value), $\min_n \mathcal{I}_n$ is positive.

To see that $\mathcal{I}_N \geq C\delta^2$, define in each finitary structure the stopping time

$$n := \min \{ i \mid |S_i| > \delta \}.$$

Again using the first equality in (1) we obtain that in each finitary structure

$$\mathcal{I}_N - 1 = \mathcal{I}_n - 1 = CS_n^2 - C \sum_{i=0}^{n-1} (dS_i)^2 > C\delta^2 - C \sum_{i=0}^{N-1} (dS_i)^2;$$

it remains to remember that the last subtrahend is infinitely small and, therefore, smaller than 1.

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