A game-theoretic derivation of the $\sqrt{dt}$ effect

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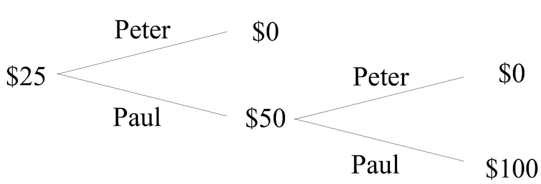

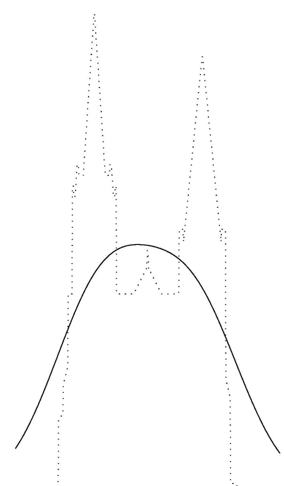

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Abstract

We study the origins of the $\sqrt{dt}$ effect in finance and SDE. In particular, we show, in the game-theoretic framework, that market volatility is a consequence of the absence of riskless opportunities for making money and that too high volatility is also incompatible with such opportunities. More precisely, riskless opportunities for making money arise whenever a traded security has fractal dimension below or above that of the Brownian motion and its price is not almost constant and does not become extremely large. This is a simple observation known in the measure-theoretic mathematical finance. At the end of the article we also consider the case of non-zero interest rate.

This version of the article was essentially written in March 2005 but remains a working paper.

Contents

1 Introduction

The main result of this article is that high market volatility is a consequence of the absence of riskless opportunities for making money. Versions of this proposition were proven within the standard continuous-time framework by Rogers (1997) (see also the references therein), Delbaen and Schachermayer (1994), etc.

In §2 we prove a simple result using nonstandard analysis saying that if a traded security is not sufficiently volatile and not too close to being a constant, this can be used for making money without risk; in the appendix to this article we explain how the informal language of §2 can be replaced by a formal argument using the ultraproduct construction described in Shafer and Vovk (2001). In the following §3 and §4 we give messier finitary forms of this result in a realistic, discrete-time setting. Results of our preliminary empirical studies are reported in §5.

In §6, we remove the assumption of zero interest rate. Our proof techniques are elementary and well-known; see, e.g., Cheridito (2001, 2002). (Although the techniques are general, the results are typically stated for very narrow classes of processes: fractional Brownian motion with drift and exponential fractional Brownian motion with drift in Cheridito 2001, 2002.)

In §8 we briefly discuss a modification of the Market Protocol of §2 that allows more natural statements of the results of §2.

2 Continuous-time result in the financial protocol

We use the notation of Shafer and Vovk (2001). In particular, $\Delta f_n := f_n - f_{n-1}$, while $df_n := f_{n+1} - f_n$. The basic framework is that of Chapter 11: the interval T is split into an infinitely large number N of subintervals etc.

THE MARKET PROTOCOL
Players: Investor, Market
Protocol:
 $\mathcal{I}_0 := 1$.
 Market announces $S_0 \in \mathbb{R}$.
 FOR $n = 1, 2, \ldots, N$:
  Investor announces $M_n \in \mathbb{R}$.
  Market announces $S_n \in \mathbb{R}$.
  $\mathcal{I}_n := \mathcal{I}_{n-1} + M_n \Delta S_n$.
Additional Constraint on Market: Market must ensure that S is continuous.

The definition of zero game-theoretic probability is given on pp. 340–341 of Shafer and Vovk (2001): an event E has zero game-theoretic probability if for any K there exists a strategy that, when started with 1, does not risk bankruptcy and finishes with capital at least K when E happens.

We start with a result showing that too low volatility gives opportunities for making money.

Theorem 1 *For any $\delta > 0$, the event*

$$\operatorname{vex} S < 2 \ \& \ \sup_t |S(t) - S(0)| > \delta$$

has game-theoretic probability zero.

The condition $\operatorname{vex} S < 2$ means that S is less volatile than the Brownian motion and $\sup_t |S(t) - S(0)| > \delta$ means that S should not be almost constant.

Proof This proof is a simple modification of Example 3 in Shiryaev (1999), p. 658, and a proof in Cheridito (2001, 2002). It is given in the usual style of Shafer and Vovk (2001); in the appendix we will provide additional details.

Assume, without loss of generality, that $S(0) = 0$ (if this is not true, replace $S(t)$ by $S(t) - S(0)$). Consider the strategy $M_n := 2CS_n$, where C is a large positive constant. With our usual notation $df_n := f_{n+1} - f_n$, we have

$$d\mathcal{I}_n = 2CS_n dS_n = C\left(d(S_n^2) - (dS_n)^2\right)$$

and, therefore,

$$\mathcal{I}_n - \mathcal{I}_0 = CS_n^2 - C\sum_{i=0}^{n-1}(dS_i)^2 \approx CS_n^2. \tag{1}$$

If this strategy starts with 1, the capital at each step n will be nonnegative. Stopping playing at the first step when $|S_n| > \delta$, we make sure that $\mathcal{I}_N \geq C\delta^2$, which can be made arbitrarily large by taking a large C. ∎

As the proof shows, the condition $\operatorname{vex} S < 2$ of the theorem can be replaced by the weaker $\mathbf{var}_S(2) = 0$.

Now we complement Theorem 1 with a result dealing with too high volatility.

Theorem 2 *For any $D > 0$, the event*

$$\operatorname{vex} S > 2 \ \& \ \sup_t |S(t) - S(0)| < D \tag{2}$$

has game-theoretic probability zero.

Proof This proof is a simple modification of a proof in Cheridito (2001, 2002). We again assume $S(0) = 0$.

Consider the strategy $M_n := -2D^{-2}S_n$. Now we have

$$d\mathcal{I}_n = -2D^{-2}S_n dS_n = D^{-2}\left((dS_n)^2 - d(S_n^2)\right)$$

and, therefore,

$$\mathcal{I}_n - \mathcal{I}_0 = D^{-2}\sum_{i=0}^{n-1}(dS_i)^2 - D^{-2}S_n^2 \geq D^{-2}\sum_{i=0}^{n-1}(dS_i)^2 - 1 \tag{3}$$

before $|S_n|$ reaches D. If this strategy starts with 1 and stops playing as soon as S_n reaches D, the capital at each step n will be nonnegative and, if event (2) occurs, $\mathcal{I}_N \geq D^{-2}\,\mathbf{var}_S(2)$ will be infinitely large. ∎

As before, the condition $\operatorname{vex} S > 2$ can be replaced by $\mathbf{var}_S(2) = \infty$.

If S is a stock price, it cannot become negative, which allows us to strengthen the conclusion of Theorem 2.

Corollary 1 *The event* $\operatorname{vex} S > 2$ *has game-theoretic probability zero (provided* $S \geq 0$*).*

Proof Let K be the constant from the definition of zero game-theoretic probability (p. 1). The required strategy is the 50/50 mixture of the following 2 strategies: the strategy of Theorem 2 corresponding to $D := 2K$ and the buy-and-hold strategy that recommends buying 1 share of S at the outset. If $\sup_t |S(t) - S(0)| < 2K$, the first strategy will make Investor rich; otherwise, the second will. ∎

3 Absolute finitary results

The protocol for this section is:

THE ABSOLUTE MARKET PROTOCOL
Players: Investor, Market
Protocol:
$\mathcal{I}_0 := 1$.
Market announces $S_0 \in \mathbb{R}$.
FOR $n = 1, 2, \ldots, N$:
Investor announces $M_n \in \mathbb{R}$.
Market announces $S_n \in \mathbb{R}$.
$\mathcal{I}_n := \mathcal{I}_{n-1} + M_n \Delta S_n$.

Now N is a usual positive integer number and there are no *a priori* constraints on Market. The following two results are the "absolute" finitary versions of Theorems 1 and 2, respectively.

Theorem 3 *Let* ϵ *and* δ *be two positive numbers. If Market is required to satisfy*

$$\sum_{i=1}^{N} (\Delta S_i)^2 \leq \epsilon,$$

the game-theoretic probability of the event

$$\max_{n=1,\ldots,N} |S_n - S_0| \geq \delta \tag{4}$$

is at most ϵ/δ^2*.*

Proof Assume, without loss of generality, that $S_0 = 0$ (replace S_n by $S_n - S_0$ if not). Take the same strategy $M_n := 2CS_n$ as in Theorem 1, but now $C = 1/\epsilon$. From (1) we obtain

$$\mathcal{I}_n - \mathcal{I}_0 = \frac{1}{\epsilon} S_n^2 - \frac{1}{\epsilon} \sum_{i=0}^{n-1} (dS_i)^2 \geq \frac{1}{\epsilon} S_n^2 - 1,$$

i.e., if the strategy starts with 1,

$$\mathcal{I}_n \geq \frac{1}{\epsilon} S_n^2.$$

This shows that $\mathcal{I}_n$ is never negative; stopping at the step n where $|S_n| \geq \delta$, we make sure that $\mathcal{I}_N \geq \delta^2/\epsilon$ when (4) happens. ∎

Theorem 4 *Let ϵ and D be two positive numbers. If Market is required to satisfy*

$$\max_{n=1,\ldots,N} |S_n - S_0| \leq D,$$

the upper game-theoretic probability of the event

$$\sum_{i=1}^{N} (\Delta S_i)^2 \geq \frac{D^2}{\epsilon} \tag{5}$$

is at most ϵ.

Proof Assume, without loss of generality, that $S_0 = 0$ (replace S_n by $S_n - S_0$ if not). Take the same strategy $M_n := -2D^{-2}S_n$ as in Theorem 2. From (3) we can see that $\mathcal{I}_n$ is never negative and that

$$\mathcal{I}_N = D^{-2} \sum_{i=1}^{N} (\Delta S_i)^2 \geq \frac{1}{\epsilon}$$

when the event (5) happens. ∎

4 Relative finitary result

Now we change our protocol to:

THE RELATIVE MARKET PROTOCOL
Players: Investor, Market
Protocol:
  $\mathcal{I}_0 := 1$.
  Market announces $S_0 > 0$.
  FOR $n = 1, 2, \ldots, N$:
    Investor announces $M_n \in \mathbb{R}$.
    Market announces $S_n > 0$.
    $\mathcal{I}_n := \mathcal{I}_{n-1} + M_n \Delta S_n$.

As in the previous section, N is a standard positive integer number. Define a nonnegative function β by

$$\frac{1}{2}\beta(x) = x - \ln(1 + x);$$

so for small $|x|$, $\beta(x)$ behaves as x^2. The following result is the "relative" finitary version of Theorem 1; it uses the versions

$$\sum_{i=0}^{N-1} (d \ln S_i)^2$$

and

$$\sum_{i=0}^{N-1} \beta\left(\frac{dS_i}{S_i}\right)$$

of the 2-variation

$$\sum_{i=0}^{N-1} \left(\frac{dS_i}{S_i}\right)^2$$

of S. (We refrain from giving a similar version of Theorem 2: such a version would be less interesting from the empirical point of view, because, as explained in the following section, the usual expectation is that $\mathbb{H} > 1/2$.)

Theorem 5 *Let ϵ, δ and γ be three positive numbers. If Market is required to satisfy*

$$\sum_{i=0}^{N-1} (d \ln S_i)^2 \le \epsilon, \quad \sum_{i=0}^{N-1} \beta\left(\frac{dS_i}{S_i}\right) \le \epsilon$$

and

$$\min_n \ln S_n \ge -\gamma,$$

the game-theoretic probability of the event

$$\max_{n=1,\ldots,N} |\ln(S_n/S_0)| \ge \delta \tag{6}$$

is at most $(1+\gamma)\epsilon/\delta^2$.

Proof Assume, without loss of generality, that $S_0 = 1$ (replace S_n by S_n/S_0 if not). Since the proof is now slightly more complicated than that in the previous section, we first outline its idea. Roughly speaking, our goal will be to maintain $\mathcal{I}_n$ close to $(\ln S_n)^2$ (in the previous sections it was to maintain $\mathcal{I}_n$ close to $(S_n)^2$). To find a strategy that will achieve this, we notice that

$$\begin{aligned} d\left(\ln^2 S_n\right) &= (2 \ln S_n)(d \ln S_n) + (d \ln S_n)^2 && (7) \\ &= (2 \ln S_n) \ln\left(1 + \frac{dS_n}{S_n}\right) + (d \ln S_n)^2 && (8) \\ &= (2 \ln S_n)\frac{dS_n}{S_n} - (\ln S_n)\beta\left(\frac{dS_n}{S_n}\right) + (d \ln S_n)^2. && (9) \end{aligned}$$

We can see that a suitable strategy is

$$M_n := 2C \frac{\ln S_n}{S_n}$$

for some C (chosen so that to make sure that the capital process is nonnegative; eventually we will take $C = 1/((1+\gamma)\epsilon)$). Expressing $2(\ln S_n)(dS_n)/S_n$ from the equality between the extreme terms of the chain (7)–(9), we obtain for the strategy M_n:

$$d\mathcal{I}_n = 2C\frac{\ln S_n}{S_n}dS_n = Cd\ln^2 S_n + C\ln S_n\beta\left(\frac{dS_n}{S_n}\right) - C(d\ln S_n)^2;$$

therefore,

$$\begin{aligned} \mathcal{I}_n - \mathcal{I}_0 &= C\ln^2 S_n + C\sum_{i=0}^{n-1}(\ln S_i)\beta\left(\frac{dS_i}{S_i}\right) - C\sum_{i=0}^{n-1}(d\ln S_i)^2 && (10)\\ &\geq C\ln^2 S_n - C\gamma\epsilon - C\epsilon. && (11) \end{aligned}$$

Starting from $\mathcal{I}_0 = 1$, it is safe to take $C := 1/((1+\gamma)\epsilon)$ (this removes possibility of bankruptcy), in which case (10)–(11) becomes

$$\mathcal{I}_n \geq \frac{1}{(1+\gamma)\epsilon}\ln^2 S_n.$$

Stopping at the step n with $|\ln S_n| \geq \delta$ ensures $\mathcal{I}_N \geq \delta^2/((1+\gamma)\epsilon)$ when (6) happens. ∎

5 Empirical studies

The empirical studies reported in this section are closely connected to the so called $\mathcal{R}/\mathcal{S}$-analysis (see Shiryaev 1999, §4a). The results we report here assume zero interest rate, and so are of limited interest; further empirical studies are needed.

First we consider the absolute setting, although the usual definitions as given in Shiryaev (1999) are "relative". Denote

$$\mathcal{R}_N^{\text{abs}} := \max_{i=1,\dots,N}|S_n - S_0|, \quad \left(\mathcal{S}_N^{\text{abs}}\right)^2 := \frac{1}{N}\sum_{i=1}^{N}(\Delta S_i)^2.$$

Suppose that we believe, for some reason, that we are going to have $\mathcal{S}_N^{\text{abs}} \leq \sigma$ and $\mathcal{R}_N^{\text{abs}} \geq \delta$; therefore, δ plays the same role as in Theorem 3 and σ plays the role of $\sqrt{\epsilon/N}$. So from Theorem 3 we obtain that we will be able to multiply our capital $\delta^2/\epsilon = (\delta/\sigma)^2/N$-fold. For another variant of the definitions of $\mathcal{R}_N$ and $\mathcal{S}_N$ (as given in Shiryaev 1999, (14) on p. 371; see below) one usually has

$$\frac{\mathcal{R}_N}{\mathcal{S}_N} \sim cN^{\mathbb{H}}$$

with $\mathbb{H}$ considerably larger than 1/2. If our guesses δ and σ are not too far off, we can hope to increase our initial capital by a factor of order $N^{2\mathbb{H}-1}$.

In the "relative" setup, define

$$\mathcal{R}_N^{\text{rel}} := \max_{n=1,\dots,N} \left| \ln \frac{S_n}{S_0} \right|, \quad \left(\mathcal{S}_N^{\text{abs}}\right)^2 := \frac{1}{N} \left(\sum_{i=0}^{N-1} \beta\left(\frac{dS_i}{S_i}\right) \bigvee \sum_{i=0}^{N-1} (d \ln S_i)^2 \right).$$

If we believe that we are going to have $\mathcal{S}_N^{\text{rel}} \le \sigma$ and $\mathcal{R}_N^{\text{rel}} \ge \delta$, we obtain from Theorem 5 that we will be able to multiply our capital by a factor of

$$\frac{\delta^2}{(1+\gamma)\epsilon} = \frac{\delta^2}{(1+\gamma)\sigma^2 N}$$

(where $\epsilon := \sigma^2 N$); if one has

$$\frac{\mathcal{R}_N}{\mathcal{S}_N} \sim c N^{\mathbb{H}}$$

and our guesses δ and σ are not too far off, we can again hope to increase our initial capital by a factor of order

$$N^{2\mathbb{H}-1}. \tag{12}$$

Some experimental results are given in Shiryaev (1999, §4.4), but we cannot use them directly, since the standard definitions of $\mathcal{R}/\mathcal{S}$ analysis are different from ours (the main difference being that the standard definitions are centered). Those results, however, suggest that typically $\mathbb{H} > 0.5$, which was why we concentrate on this case in our discrete-time analysis and empirical studies.

In our experiments we consider, instead of $\mathcal{R}_N^{\text{abs}}$ and $\mathcal{R}_N^{\text{rel}}$, $|S_N - S_0|$ and $|\ln(S_N/S_0)|$, respectively, the rationale being that security prices typically increase. This frees us from the need to guess the value of δ in advance. Our results are summarized in Tables 1 and 2.

In Table 1 we list the 19 securities for which we conducted experiments. The number N is the number of trading periods (days, month, or years).

The numbers given in Table 2 are defined as follows:

$$\text{abs factor} := \frac{(S_N - S_0)^2}{\sum_{i=0}^{N-1} (dS_i)^2}$$

and

$$\text{rel factor} := \frac{\left(\ln \frac{S_N}{S_0}\right)^2}{(1-\min)\left(\sum_{i=0}^{N-1} (d\ln S_i)^2 \bigvee \sum_{i=0}^{N-1} \beta\left(\frac{dS_i}{S_i}\right)\right)},$$

where

$$\min := \min_n \ln \frac{S_n}{S_0}.$$

To judge the magnitude of abs factor and rel factor we also give the factor by which the value of the security increases (the column "security") and the factor by which the value of an index (S&P500) increases (the column "index") over the same time period.

Security and frequency	Code	Time Period	N
Microsoft stock daily	msft d	13/03/1986–21/09/2001	3672
IBM stock daily	ibm d	02/01/1962–21/09/2000	9749
S&P500 daily	spc d	04/01/1960–21/09/2000	10,254
Microsoft stock monthly	msft m	March 1986–June 2001	184
IBM stock monthly	ibm m	January 1962–June 2001	474
General Electric stock monthly	ge m	January 1962–June 2001	474
Boeing stock monthly	ba m	January 1970–June 2001	378
Du Pont (E.I.) de Nemours stock monthly	dd m	January 1970–June 2001	378
Consolidated Edison stock monthly	ed m	January 1970–June 2001	378
Eastman Kodak stock monthly	ek m	January 1970–June 2001	378
General Motors stock monthly	gm m	January 1970–June 2001	378
Procter and Gamble stock monthly	pg m	January 1970–June 2001	378
Sears/Roebuck stock monthly	s m	January 1970–June 2001	378
AT&T stock monthly	t m	January 1970–June 2001	378
Texaco stock monthly	tx m	January 1970–June 2001	378
US T-bill monthly	us m	January 1871–June 2001	1566
S&P500 Total Returns monthly	sp m	January 1871–June 2001	1566
US T-bill yearly	us a	1871–2000	130
S&P500 Total Returns yearly	sp a	1871–2001	130

Table 1: The 19 securities used in our experiments. Dates are given in the format dd/mm/yyyy.

As we already mentioned, our experiments implicitly assume zero interest rate, but the results they give are roughly of the same order of magnitude as those implied by the table on p. 376 of Shiryaev (1999). Line 1 of that table can be interpreted (ignoring the facts that centering is not the same thing as discounting and that DJIA cannot be reproduced by a trading strategy) as saying that our initial capital can be increased by a factor of roughly

$$12,500^{2\times 0.59-1} \approx 5.46$$

in 12,500 days since 1888.

6 Non-zero interest rate

Our protocols implicitly assume that the interest rate is zero. In this section we remove this restriction. Our protocol now involves not only security S but also another security B (e.g., a bank account). Their prices are assumed positive.

THE MARKET PROTOCOL
Players: Investor, Market
Protocol:
$\mathcal{I}_0 := 1$.
Market announces $S_0 > 0$ and $B_0 > 0$.

code	abs factor	rel factor	index	security	min
msft d	1.32	13.8	6.13	330	−0.0736
ibm d	2.62	1.94	20.8	16.0	−0.626
spc d	8.25	10.7	24.2	24.2	−0.138
msft m	0.930	12.6	7.59	382	0
ibm m	2.08	2.22	69.4	15.8	−0.469
ge m	4.57	6.99	69.4	63.4	−0.232
ba m	2.35	3.25	43.5	75.5	−0.620
dd m	1.66	6.72	43.5	33.0	0
ed m	4.21	3.76	43.5	58.0	−1.19
ek m	0.638	0.994	43.5	5.06	−0.441
gm m	1.41	2.24	43.5	13.8	−0.462
pg m	1.20	4.88	43.5	18.6	−0.166
s m	0.0331	0.0223	43.5	1.37	−0.723
t m	0.0293	0.0159	43.5	0.702	−1.13
tx m	4.66	7.97	43.5	36.7	0
us m	3.87	1081	91600	282	0
sp m	6.99	32.9	91,600	91,600	−0.0877
us a	32.2	93	87600	261	0
sp a	6.57	23.9	87,600	87,600	−0.00663

Table 2: Empirical results related to Theorems 3 and 5.

FOR $n = 1, 2, \dots, N$:
 Investor announces $M_n \in \mathbb{R}$.
 Market announces $S_n > 0$ and $B_n > 0$.
 $\mathcal{I}_n := (\mathcal{I}_{n-1} - M_n S_{n-1}) \frac{B_n}{B_{n-1}} + M_n S_n$.
Additional Constraint on Market: Market must ensure that S and B are continuous.

(Cf. the protocol and its analysis on p. 296 of Shafer and Vovk 2001.) Intuitively, at step n Investor buys M_n units of S and invests the remaining money in B, which can be a money market account, a bond, or any other security with nonnegative prices. The protocol of §2 corresponds to a constant B_n.

Re-expressing Investor's capital and the price of S in the *numéraire* B_n, we obtain

$$\mathcal{I}_n^\dagger := \mathcal{I}_n / B_n, \quad S_n^\dagger := S_n / B_n.$$

It is easy to see that

$$\mathcal{I}_n^\dagger := \mathcal{I}_{n-1}^\dagger - M_n S_{n-1}^\dagger + M_n S_n^\dagger,$$

which is exactly the expression that we had in §2, only with the daggers added. Therefore, we can restate all results of §2 for the current protocol. For example, Theorem 8 implies:

Theorem 6 *The event*

$$\mathrm{vex}(S/B) = 2 \text{ or } S/B \approx \mathrm{const}$$

is full.

We have an interesting all-or-nothing phenomenon: either two securities are proportional or their ratio behaves stochastically.

7 Continuous-time result in the drift-SDE protocol

In this section we consider a slightly more general protocol (see Chapter 14 of Shafer and Vovk 2001):

THE DRIFT-SDE PROTOCOL
Players: Forecaster, Skeptic, Reality
Protocol:
  $\mathcal{I}_0 := 1$.
  Reality announces $S_0 \in \mathbb{R}$; $T_0 := S_0$.
  FOR $n = 1, 2, \dots, N$:
    Forecaster announces $m_n \in \mathbb{R}$; $T_n := T_{n-1} + m_n$.
    Skeptic announces $M_n \in \mathbb{R}$.
    Reality announces $S_n \in \mathbb{R}$; $x_n := \Delta S_n$.
    $\mathcal{I}_n := \mathcal{I}_{n-1} + M_n(x_n - m_n)$.
Additional Constraint on Market: Market must ensure that S and T are continuous.

The main differences from the Market Protocol are that: Market becomes Reality; Investor becomes Skeptic; a new player, Forecaster, is introduced, who announces at each trial his expectation of the increment x_n to be chosen by Reality (the Market Protocol corresponds to the case where m_n is always 0). The definition of game-theoretic upper probability is unchanged.

In Chapter 14 of Shafer and Vovk (2001) we describe Diffusion Protocol 1, a game-theoretic counterpart of the standard measure-theoretic SDE

$$dS(t) = \mu(S(t), t)dt + \sigma(S(t), t)dW(t);$$

this equation is modeled by Forecaster choosing the moves

$$m_n := \mu(S_{n-1}, ndt)dt \tag{13}$$

(the *drift move*) and

$$v_n := \sigma^2(S_{n-1}, ndt)dt$$

(the *volatility move*). Already Diffusion Protocol 1 provides a flexible alternative to the usual measure-theoretic approach to SDE; we believe that it would be very beneficial to translate the standard theory of SDE to the game-theoretic framework liberating measure-theoretic results of unnecessary assumptions. But we can also do more radical things considering much weaker protocols than Diffusion Protocol 1. Diffusion Protocol 2 in Shafer and Vovk (2001) drops Forecaster's drift move altogether; it turns out (Shafer and Vovk 2001, Theorem 14.1) that the Black-Scholes formula can be proven in Diffusion Protocol 2. (It is well known that in the measure-theoretic framework the Black-Scholes formula does not depend on drift, but still there is no way to drop the assumption of existence of drift.) In this section we relax Diffusion Protocol 1 in a different way: now we drop Forecaster's volatility move. We will see that this will not prevent us from proving the $\sqrt{dt}$ effect.

First we motivate the conditions of our theorem. According to (13), m_n has the order of magnitude dt; in the game-theoretic framework we also expect that the drift process $T(t)$ will be much more stable than the process $S(t)$ itself. Therefore, one of our conditions will be that $\sum_n m_n^2$ is infinitely small.

Theorem 7 *For any $\delta > 0$ and $D > 0$, the event*

$$\left.\begin{array}{c}\sum_{n=1}^N m_n^2 \approx 0\\ \delta < \sup_t |S(t) - T(t)| < D\end{array}\right\} \Longrightarrow \sum_{n=1}^N x_n^2 \text{ is appreciable}$$

has lower game-theoretic probability one.

Proof Set $x'_n := x_n - m_n$. It is easy to see from the arguments of §2 that the event

$$\left.\begin{array}{c}\sum_{n=1}^N m_n^2 \approx 0\\ \delta < \sup_t |S(t) - T(t)| < D\end{array}\right\} \Longrightarrow \sum_{n=1}^N (x'_n)^2 \text{ is appreciable}$$

has lower game-theoretic probability one (even if the condition $\sum_{n=1}^N m_n^2 \approx 0$ is dropped). The fact that

$$\left.\begin{array}{c}\sum_{n=1}^N m_n^2 \approx 0\\ \delta < \sup_t |S(t) - T(t)| < D\end{array}\right\} \Longrightarrow \sum_{n=1}^N x_n^2 \text{ is limited}$$

has lower game-theoretic probability one now follows from the closeness of L^2 under addition; more specifically, from

$$x_n^2 = (m_n + x'_n)^2 \le 2\left(m_n^2 + (x'_n)^2\right).$$

Therefore, we only need to prove that

$$\left.\begin{array}{c}\sum_{n=1}^N m_n^2 \approx 0\\ \delta < \sup_t |S(t) - T(t)| < D\end{array}\right\} \Longrightarrow \sum_{n=1}^N x_n^2 \text{ is not infinitesimal}$$

has lower game-theoretic probability one. In other words, our goal is to prove that the event

$$\sum_{n=1}^N m_n^2 \approx 0 \;\;\&\;\; \delta < \sup_t |S(t) - T(t)| < D \;\;\&\;\; \sum_{n=1}^N x_n^2 \approx 0 \qquad (14)$$

has zero upper game-theoretic probability.

According to (1), we have, for some strategy S_1 for Skeptic,

$$\mathcal{I}_N^{S_1} - \mathcal{I}_0^{S_1} = C(S_N - T_N)^2 - C\sum_{i=1}^{N}(x_i')^2.$$

Since

$$x_i^2 = (x_i' + m_i)^2 = (x_i')^2 + 2x_i' m_i + m_i^2,$$

we can rewrite this equality as

$$\mathcal{I}_N^{S_1} - \mathcal{I}_0^{S_1} = C(S_N - T_N)^2 - C\sum_{i=1}^{N} x_i^2 + C\sum_{i=1}^{N} m_i^2 + \mathcal{I}_N^{S_2} - \mathcal{I}_0^{S_2},$$

where S_2 is Skeptic's strategy that recommends move $2Cm_i$ at trial i. Therefore, there is Skeptic's strategy that ensures

$$\mathcal{I}_N - \mathcal{I}_0 = C(S_N - T_N)^2 - C\sum_{i=1}^{N} x_i^2 + C\sum_{i=1}^{N} m_i^2,$$

and we can take $\mathcal{I}_0$ to be 1. On the event (14) this strategy (if stopped at the first moment that $|S(t) - T(t)| > \delta$) will attain at least a capital of $C\delta^2$, which can be made as large as we wish by choosing a large C. ∎

8 A modified Market Protocol

To state Theorems 1 and 2 in a nicer way (avoiding the ϵ and D), we change the Market Protocol in the following way. The two parameters of the Market Protocol were T, the time horizon, and N, the infinite number of subintervals into which the interval $[0, T]$ was split. Now we allow T to be an infinitely large positive number (still requiring $dt := T/N$ to be infinitesimal) and add another parameter, an infinitely small positive number ϵ. (Of course, T can stay limited if we wish.) The Additional Constraint on Market is now changed to "Market must ensure that $\sup |\Delta S| \le \epsilon$". The upper probability P in this protocol is defined by the formula

$$P(E) := \inf \left\{ \mathcal{I}^S(\Box) \mid \inf_{0 \le t \le T} \mathcal{I}^S(t) \ge 0 \text{ everywhere}, \ \mathcal{I}^S(T) \ge 1 \text{ inside } E \right\},$$

where S ranges over (internal) strategies; the expressions such as "almost certain" refer to this upper (and the corresponding lower) probability. Remember that a hyperreal number t is *appreciable* if $a < |t| < b$ for some positive real a and b (i.e., if it is neither unlimited nor infinitesimal).

Theorem 8 *It is almost certain that*

$$\sup_t |S(t) - S(0)| \text{ is appreciable} \implies \operatorname{vex} S = 2.$$

More precisely,

$$\operatorname{vex} S < 2 \Longrightarrow \sup_t |S(t) - S(0)| \text{ is infinitesimal} \tag{15}$$

and

$$\operatorname{vex} S > 2 \Longrightarrow \sup_t |S(t) - S(0)| \text{ is unlimited.} \tag{16}$$

Proof Of course, the proof is a modification of the proofs of Theorems 1 (for (15)) and 2 (for (16)); we again assume $S(0) = 0$.

First we prove (15). As before, we consider the strategies $M_n^{(C)} := 2CS_n$ starting from the initial capital 1, with the only difference that the strategy stops playing (i.e., starts choosing the move 0) as soon as $C\sum_{i=0}^{n-1}(dS_i)^2$ reaches the value $1 - C\epsilon^2$ (in particular, the strategy never plays if $C\epsilon^2 \geq 1$; this stopping rule ensures that the strategy never goes bankrupt) or $|S_n| > C^{-1/2}$ (this condition replaces $|S_n| > \delta$), whichever happens earlier. Now we can combine these strategies into

$$M_n := \sum_{m=1}^{\infty} 2^{-m} M_n^{(2^m)}$$

(we do not have any problems of convergence since for each standard $\epsilon > 0$ only finitely many strategies $M_n^{(2^m)}$ will ever play). It is clear that this strategy will ensure an unlimited final capital $\mathcal{I}_N$.

It remains to prove (16). Consider the strategies $M_n^{(D)} := -2D^{-1/2}S_n$ starting from the initial capital 1, with the only difference that the strategy stops playing as soon as $D^{-2}S_n^2$ reaches the value $1 - D^{-2}(2S_n\epsilon + \epsilon^2)$. This way we make sure that the strategy never goes bankrupt. Combining, as before, the strategies $M_n^{(D)}$ into

$$M_n := \sum_{m=1}^{\infty} 2^{-m} M_n^{(2^m)}$$

(the convergence follows from $\sum_{m=1}^{\infty} 2^{-3m} < \infty$), we can see that the combined strategy will ensure an unlimited final capital $\mathcal{I}_N$. ∎

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A Continuous games

This appendix contains a partial account of the concepts from nonstandard analysis used in this article. (It was intended as an improvement over Appendix 11.5 of Shafer and Vovk (2001).)

The continuous games that we consider in the main part of the article are ultraproducts of discrete games. We will first explain informally how such ultraproducts are formed. For a formal exposition of the concept of an ultraproduct, the reader may consult Eklof (1977) or the classical article by Jerzy Łoś (1955).

In general, an ultraproduct is formed from a sequence $\mathcal{O}_1, \mathcal{O}_2, \dots$ of similar mathematical structures, perhaps identical or perhaps increasing in size. We remain informal by not saying what we mean by "similar", but the idea is that certain statements have a meaning in each of the $\mathcal{O}_n$. A statement that two objects are related in a certain way, for example, might be interpreted in $\mathcal{O}_n$ as $R_n(x_n, y_n)$, where x_n and y_n are objects in $\mathcal{O}_n$ and R_n is a binary relation in $\mathcal{O}_n$. Such a statement should also have a reference $R(x, y)$ in the ultraproduct. Intuitively,

- R is the sequence $R_1, R_2, \dots,$
- x is the sequence $x_1, x_2, \dots,$
- y is the sequence $y_1, y_2, \dots,$ and

- $R(x, y)$ holds if $R_n(x_n, y_n)$ holds for most n.

To make "most" precise, we choose a nontrivial ultrafilter in $\mathbb{N}$, the set of natural numbers (positive integers). A nontrivial ultrafilter $\mathcal{U}$ in $\mathbb{N}$ is a set of subsets of $\mathbb{N}$ that has, inter alia, the property that whenever we partition $\mathbb{N}$ into two sets, exactly one of the two sets is in $\mathcal{U}$. We say a relation holds for most n if the set of n for which it holds is in $\mathcal{U}$.

To strengthen this explanation, we now review the concept of an ultrafilter and provide two examples of an ultraproduct: (i) the hyperreals, and (ii) a simple continuous game.

A.1 Ultrafilters

An *ultrafilter* in $\mathbb{N}$ is a family $\mathcal{U}$ of subsets of $\mathbb{N}$ such that

1. $\mathbb{N} \in \mathcal{U}$ and $\emptyset \notin \mathcal{U}$,
2. if $A \in \mathcal{U}$ and $A \subseteq B \subseteq \mathbb{N}$, then $B \in \mathcal{U}$,
3. if $A \in \mathcal{U}$ and $B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$, and
4. if $A \subseteq \mathbb{N}$, then either $A \in \mathcal{U}$ or $\mathbb{N} \setminus A \in \mathcal{U}$.

(The first three properties define a *filter*.) An ultrafilter $\mathcal{U}$ is *nontrivial* if it does not contain a set consisting of a single integer; this implies that all the sets in $\mathcal{U}$ are infinite. It follows from the axiom of choice that a nontrivial ultrafilter exists. We fix a nontrivial ultrafilter $\mathcal{U}$.

We say that a property of natural numbers holds for *most* natural numbers (or for most k, as we will say for brevity) if the set of natural numbers for which it holds is in $\mathcal{U}$; Condition 2 of the definition justifies this usage. It follows from Condition 4 that for any property A, either A holds for most k or else the negation of A holds for most k. It follows from Conditions 1 and 3 that A and its negation cannot both hold for most k.

A.2 The hyperreals

As a first example of an ultraproduct, we construct the hyperreals, as they are usually constructed in nonstandard analysis Goldblatt (1998). In this case, the objects $\mathcal{O}_n$ are all identical—each is a copy of the real numbers, together with the usual operations and relations associated with them.

As a first approximation, a *hyperreal number a* is a sequence $\left[a^{(1)}a^{(2)}\dots\right]$ of real numbers. Sometimes we abbreviate $\left[a^{(1)}a^{(2)}\dots\right]$ to $\left[a^{(k)}\right]$. Operations (addition, multiplication, etc.) over hyperreals are defined term by term. For example,

$$\left[a^{(1)}a^{(2)}\dots\right] + \left[b^{(1)}b^{(2)}\dots\right] := \left[\left(a^{(1)} + b^{(1)}\right)\left(a^{(2)} + b^{(2)}\right)\dots\right].$$

Relations (equals, greater than, etc.) are extended to the hyperreals by voting. For example, $[a^{(1)}a^{(2)}\dots] \leq [b^{(1)}b^{(2)}\dots]$ if $a^{(k)} \leq b^{(k)}$ for most k. For all

$a, b \in {}^*\mathbb{R}$ one and only one of the following three possibilities holds: $a < b$, $a = b$, or $a > b$.

Perhaps we should dwell for a moment on the fact that a hyperreal number $a = \left[a^{(1)}a^{(2)}\dots\right]$ is always below, equal to, or above another hyperreal number $b = \left[b^{(1)}b^{(2)}\dots\right]$: $a < b$, $a = b$, or $a > b$. Obviously some of the $a^{(k)}$ can be above $b^{(k)}$, some equal to $b^{(k)}$, and some below $b^{(k)}$. But the set of k satisfying one these three conditions is in $\mathcal{U}$ and outvotes the other two.

We do not distinguish hyperreals a and b such that $a = b$. Technically, this means that a hyperreal is an equivalence class of sequences rather than an individual sequence: $[a^{(1)}a^{(2)}\dots]$ is the equivalence class containing $a^{(1)}a^{(2)}\dots$.

We embed the real numbers in the hyperreals by identifying each real number a with $[a, a, \dots]$. For each $A \subseteq \mathbb{R}$ we denote by ${}^*A$ the set of all hyperreals $[a^{(k)}]$ with $a^{(k)} \in A$ for all k. We call ${}^*\mathbb{N}$ the *hypernaturals*.

We say that $a \in {}^*\mathbb{R}$ is *infinitesimal* if $|a| < \epsilon$ for each real $\epsilon > 0$. The only real number that qualifies as an infinitesimal by this definition is 0. We say that $a \in {}^*\mathbb{R}$ is *infinitely large* if $a > C$ for each positive integer C, and we say that $a \in {}^*\mathbb{R}$ is *finite* if $a < C$ for some positive integer C.

We write $a \approx b$ when $a - b$ is infinitesimal. For every hyperreal number $a \in {}^*\mathbb{R}$ there exists a unique standard number $\mathrm{st}(a)$ (its *standard part*) such that $a \approx b$.

The representation of the hyperreals as equivalence classes of sequences with respect to a nontrivial ultrafilter is constructive only in a relative sense, because the proof that a nontrivial ultrafilter exists is nonconstructive; no one knows how to exhibit one. However, the representation provides an intuition that helps us think about hyperreals. For example, an infinite positive integer is represented by a sequence of positive integers that increases without bound, such as $[1, 2, 4, \dots]$, and the faster it grows the larger it is.

A.3 An ultraproduct of games

Now we construct a continuous game, of the type used in this article.

In this construction, the following protocol, where n is a natural number:

Protocol:

$\mathcal{I}_0 := 1$.
Market announces $S_0 \in \mathbb{R}$.
FOR $n = 1, 2, \dots, N$:
  Investor announces $M_n \in \mathbb{R}$.
  Market announces $S_n \in \mathbb{R}$.
  $\mathcal{I}_n := \mathcal{I}_{n-1} + M_n \Delta S_n$.

Fix a positive real number T and an infinitely large positive integer N; let

$$N = [N^{(k)}] = [N^{(1)}, N^{(2)}, \dots].$$

For each natural number k, set

$$\mathbb{T}^{(k)} := \{nT/N^{(k)} \mid n = 0, 1, \dots, N^{(k)}\}.$$

To each k corresponds a "finitary framework" (which we will call the k-finitary framework), where the time interval is the finite set $\mathbb{T}^{(k)}$ rather than the infinite set $\mathbb{T}$. The "limit" (formally, ultraproduct) of these finitary frameworks will be the infinitary framework based on $\mathbb{T}$; as in the previous subsection, this "limit" is defined as follows:

- An object in the infinitary framework, such as strategy, should be defined as a family of finitary objects: for every k, an object in the k-finitary framework should be defined (cf. the definition of hyperreals in the previous subsection).
- Functionals defined on finitary objects are extended to infinitary objects term-wise, analogously to the previous subsection. (By "functionals" we mean functions of objects of complex nature, such as paths or strategies.)
- Relations (in particular, properties) are defined by voting (again as in the previous subsection).

(In nonstandard analysis such limiting infinitary structures are called hyperfinite.)

A.4 Details of the proof of Theorem 1

Let us show more formally why $\mathcal{I}_n$ is nonnegative and why $\mathcal{I}_N \geq C\delta^2$.

According to the first equality in (1), in every finitary framework we have

$$\mathcal{I}_n - 1 \geq -C \sum_{i=0}^{N-1} (dS_i)^2;$$

since the value on the right-hand side is infinitesimal (and, therefore, smaller than 1 in absolute value), $\min_n \mathcal{I}_n$ is positive.

To see that $\mathcal{I}_N \geq C\delta^2$, define in each finitary framework the stopping time

$$n := \min \{ i \mid |S_i| > \delta \} .$$

Again using the first equality in (1) we obtain that in each finitary framework

$$\mathcal{I}_N - 1 = \mathcal{I}_n - 1 = CS_n^2 - C \sum_{i=0}^{n-1} (dS_i)^2 > C\delta^2 - C \sum_{i=0}^{N-1} (dS_i)^2;$$

it remains to remember that the last subtrahend is infinitely small and, therefore, smaller than 1.