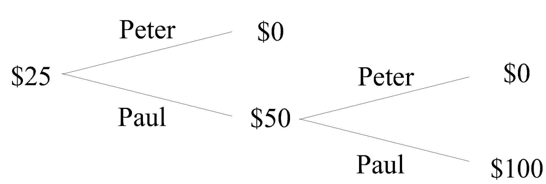
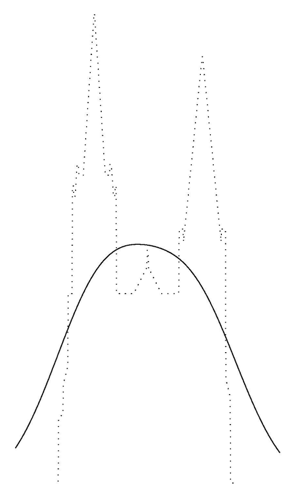
Game-Theoretic Capital Asset Pricing in Continuous Time

Vladimir Vovk and Glenn Shafer





The Game-Theoretic Probability and Finance Project

Working Paper #2

First posted December 30, 2001. Last revised February 5, 2018.

Project web site:
http://www.probabilityandfinance.com

Abstract

We derive formulas for the performance of capital assets in continuous time from an efficient market hypothesis, with no stochastic assumptions and no assumptions about the beliefs or preferences of investors. Our efficient market hypothesis says that a speculator with limited means cannot beat a particular index by a substantial factor. Our results include a formula that resembles the classical CAPM formula for the expected simple return of a security or portfolio.

This version of the article was essentially written in December 2001 but remains a working paper.

Contents

In this article, we use an efficient market hypothesis to derive formulas for the performance of capital assets in continuous time. Our efficient market hypothesis says that a speculator with limited means cannot beat a particular market index m by a substantial factor. From this hypothesis, we derive two results concerning the average returns of a security or portfolio s:

- The average simple return for s will fall short of the average simple return for m by an amount equal to the difference between the variance of m's simple returns and the covariance of m's and s's simple returns. (See the formula in Proposition 1.) This agrees with the classical capital asset pricing model, except that the classical model relates theoretical centered moments of a probability distribution, whereas our result is about empirical uncentered moments.
- The average logarithmic return for s will fall short of the average logarithmic return for m by an amount equal to one-half the variance of the difference between the simple returns. (See the formula in Proposition 2.)

These results follow from the efficient market hypothesis alone, with no stochastic assumptions and no assumptions about the beliefs or preferences of investors.

Our results are for continuous time and use nonstandard analysis. They parallel the results for discrete time (with explicit error bounds) in our article [3]. That article gives more detail on motivation and possible applications. For an alternative treatment of continuous time (more traditional and not using nonstandard analysis), see [4] and [5].

1 Definitions

Our results depend on the explicit formulation of a game. Here we begin by describing the game informally. Then, after introducing notation for the moments of s and m's returns, we state the protocol for the game formally and explain how it can be used to formalize the implications of our efficient market hypothesis.

1.1 The Basic Capital Asset Pricing Game

The capital asset pricing game has two principal players, Speculator and Market, who alternate play. On each round, Speculator decides how much of each security in the market to hold (and possibly short), and then Market determines Speculator's gain by deciding how the prices of the securities change. Allied with Market is a third player, Investor, who also invests each day. The game is a perfect-information game: each player sees the others' moves.

We assume that there are $K + 1$ securities in the market and N rounds (trading periods) in the game. We number the securities from 0 to K and the rounds from 1 to N, and we write x_n^k for the simple return on security k in round n. For simplicity, we assume that $-1 < x_n^k < \infty$ for all k and n; a security price never becomes zero. We write x_n for the vector $(x_n^0, \ldots, x_n^K)$, which lies in $(-1,\infty)^{K+1}$. Market determines the returns; x_n is his move on the nth round.

We assume that the first security, indexed by 0, is our market index m. If m is a portfolio formed from the other securities, then x_n^0 is an average of the $x_n^1, \ldots, x_n^K$, but we do not insist on this.

We write $\mathcal{M}_n$ for the capital at the end of round n resulting from investing one monetary unit in m at the beginning of the game:

$$\mathcal{M}_n := \prod_{i=1}^{n} (1 + x_i^0).$$

Thus $\mathcal{M}_N$ is the final capital resulting from this investment.

Investor begins with capital equal to one monetary unit and is allowed to redistribute his current capital across all $K+1$ securities on each round. If we write $\mathcal{G}_n$ for his capital at the end of the nth round, then

$$\mathcal{G}_n := \prod_{i=1}^{n} \sum_{k=0}^{K} g_i^k (1 + x_i^k),$$

where g_i^k is the fraction of his capital he holds in security k during the ith round. The g_i^k must sum to 1 over k, but g_i^k may be negative for a particular k (in this case Investor is selling k short). Investor's final capital is $\mathcal{G}_N$.

We call the set of all possible sequences $(g_1, x_1, \ldots, g_N, x_N)$ the *sample space* of the game, and we designate it by Ω:

$$\Omega := \left(\mathbb{R}^{K+1} \times (-1,\infty)^{K+1} \right)^N .$$

We call any subset of Ω an *event*. Any statement about Investor's returns determines an event, as does any comparison of Investor's and Market's returns.

Speculator also starts with one monetary unit and is allowed to redistribute his current capital across all $K+1$ securities on each round. We write $\mathcal{H}_n$ for his capital at the end of the nth round:

$$\mathcal{H}_n := \prod_{i=1}^{n} \sum_{k=0}^{K} h_i^k (1 + x_i^k),$$

where h_i^k is the fraction of his capital he holds in security k during the ith round. The moves by Speculator are not recorded in the sample space; they do not define events.

Our results use non-standard analysis. We assume that the number N of rounds of the game is infinitely large. The game begins at time 0. Each round takes an infinitesimal amount of time dt, and play ends at time T, which is an infinitely large positive real number: $T = N dt$. A brief summary of nonstandard analysis, sufficient for our purposes, is provided in [2], §11.5; further details can be found in [1].

1.2 Notation for Moments

Let us write s_n for Investor's simple return on round n, and m_n for the simple return of the market index m on round n:

$$s_n := \frac{\mathcal{G}_n - \mathcal{G}_{n-1}}{\mathcal{G}_{n-1}} = \sum_k g_n^k x_n^k,$$

and

$$m_n := x_n^0.$$

For every play of the game we define the following nonstandard numbers:

$$\mu_s = \frac{1}{N}\sum_{n=1}^{N}\frac{s_n}{dt} = \frac{1}{T}\sum_{n=1}^{N} s_n$$

(this is the average rate of increase in Investor's capital),

$$\sigma_s^2 = \frac{1}{N}\sum_{n=1}^{N}\frac{s_n^2}{dt} = \frac{1}{T}\sum_{n=1}^{N} s_n^2$$

(σ_s is the empirical volatility of Investor's capital),

$$\mu_m = \frac{1}{N}\sum_{n=1}^{N}\frac{m_n}{dt} = \frac{1}{T}\sum_{n=1}^{N} m_n, \quad \sigma_m^2 = \frac{1}{N}\sum_{n=1}^{N}\frac{m_n^2}{dt} = \frac{1}{T}\sum_{n=1}^{N} m_n^2$$

(analogous quantities for the index). We also set

$$\sigma_{sm} = \frac{1}{N}\sum_{n=1}^{N}\frac{s_n m_n}{dt} = \frac{1}{T}\sum_{n=1}^{N} s_n m_n,$$

$$\sigma_{s-m}^2 = \frac{1}{N}\sum_{n=1}^{N}\frac{(s_n - m_n)^2}{dt} = \frac{1}{T}\sum_{n=1}^{N}(s_n - m_n)^2$$

and

$$\lambda_s = \frac{1}{N}\sum_{n=1}^{N}\frac{\ln(1+s_n)}{dt} = \frac{1}{T}\sum_{n=1}^{N}\ln(1+s_n),$$

$$\lambda_m = \frac{1}{N}\sum_{n=1}^{N}\frac{\ln(1+m_n)}{dt} = \frac{1}{T}\sum_{n=1}^{N}\ln(1+m_n)$$

(the last two are the average logarithmic rates of growth). Notice that $\exp(\lambda_s T)$ is the total relative increase $\mathcal{G}_N$ in Investor's capital and $\exp(\lambda_m T)$ is the total relative increase $\mathcal{M}_N$ in the value of the index; therefore, λ_s and λ_m are direct measures of Investor's and the index's performance.

As a simple example, consider the case where Investor always holds one share of a security whose price S_t is generated by Market using the stochastic differential equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

where W_t is a Brownian motion. In this case, μ_s will be infinitely close to μ almost surely, and σ_s will be infinitely close to σ almost surely.

1.3 The Protocol

We can now state precisely the protocol for the game involving Investor, Market, and Speculator:

Basic Capital Asset Pricing Protocol (Basic CAP Protocol)
Players: Investor, Market, Speculator
Parameters:
  Natural number K (number of non-index securities in the market)
  Infinite natural number N (number of rounds or trading periods)
Protocol:
  $\mathcal{G}_0 := 1$.
  $\mathcal{H}_0 := 1$.
  $\mathcal{M}_0 := 1$.
  FOR $n = 1, 2, \ldots, N$:
    Investor selects $g_n \in \mathbb{R}^{K+1}$ such that $\sum_{k=0}^K g_n^k = 1$.
    Speculator selects $h_n \in \mathbb{R}^{K+1}$ such that $\sum_{k=0}^K h_n^k = 1$.
    Market selects $x_n \in (-1, \infty)^{K+1}$.
    $\mathcal{G}_n := \mathcal{G}_{n-1} \sum_{k=0}^K g_n^k (1 + x_n^k)$.
    $\mathcal{H}_n := \mathcal{H}_{n-1} \sum_{k=0}^K h_n^k (1 + x_n^k)$.
    $\mathcal{M}_n := \mathcal{M}_{n-1} (1 + x_n^0)$.
Restriction:
Market and Investor are required to make σ_s^2 and σ_m^2 finite and to make $\max_n |s_n|$ and $\max_n |m_n|$ infinitesimal.

The condition that $\max_n |s_n|$ and $\max_n |m_n|$ be infinitesimal is a continuity condition, but there is a slight complication arising from the fact that T is infinite: if T is extremely large as compared with $1/dt$, the largest of the huge number of increments might become nonnegligible. This, however, would be an extreme situation: e.g., for the usual diffusion processes the condition $\ln T \leq (dt)^{-1/2}$ is more than sufficient for the largest increment to be negligible.

1.4 Predictions from the EMH for m

We now adopt an efficient market hypothesis: Market will not allow Speculator to become very rich relative to the index m. Intuitively, this sometimes implies that a certain event A will happen. To formalize this intuition, let us say that *the efficient market hypothesis for m predicts A at level $\alpha > 0$* if Speculator has a strategy $\mathcal{S}$ in the basic CAP protocol that ensures the following: $\mathcal{H}_n \geq 0$ for $n = 1, \dots, N$ and either (1) $\mathcal{H}_N \geq \frac{1}{\alpha}\mathcal{M}_N$ or (2) $(g_1, x_1, \dots, g_N, x_N) \in A$. (It will be convenient to say that such a strategy $\mathcal{S}$ *witnesses* that the efficient market hypothesis predicts A at level α.)

For brevity, we will abbreviate "efficient market hypothesis for m" to "EMH for m". As the reader may have noticed, "EMH for m" is not a mathematical concept for us; we have not given it a precise definition. We have, however, provided a precise definition for the phrase "the EMH for m predicts A at level $\alpha > 0$".

Our confidence that Speculator will not beat the market by $\frac{1}{\alpha}$ is greater for smaller α. So a prediction of A at level α becomes more emphatic as α decreases. The most emphatic prediction arises in the limit, when the EMH for m predicts A at every level $\alpha > 0$. In this case, we say simply that *the EMH for m predicts A*.

2 Results

Proofs of the propositions in this section will be provided in §3.

2.1 Capital Asset Pricing Model

Proposition 1. *For any $\epsilon > 0$, the EMH for m predicts*

$$\left|\mu_s - \mu_m + \sigma_m^2 - \sigma_{sm}\right| < \epsilon(1 + \sigma_{s-m}^2).$$

If it is known *a priori* that $\sigma_{s-m}^2 < C$ for some positive constant C, then for every $\epsilon > 0$, the EMH for m predicts that

$$\left|\mu_s - \mu_m + \sigma_m^2 - \sigma_{sm}\right| < \epsilon.$$

This is analogous to the capital asset pricing model (CAPM) of the established theory. Remarkably, we get the result without the strong assumptions of that theory. We assume nothing about Investor's beliefs or preferences, and we do not assume that Market chooses his prices stochastically.

2.2 Theoretical Performance Deficit

The next proposition justifies calling $\sigma_{s-m}^2/2$ the "theoretical performance deficit".

Proposition 2. *For any $\epsilon > 0$, the EMH for m predicts that*

$$\left|\lambda_s - \lambda_m + \frac{1}{2}\sigma_{s-m}^2\right| < \epsilon(1 + \sigma_{s-m}^2).$$

Again if it is known *a priori* that $\sigma_{s-m}^2 < C$ for some positive constant C, then for every $\epsilon > 0$ the EMH for m predicts that

$$\left|\lambda_s - \lambda_m + \frac{1}{2}\sigma_{s-m}^2\right| < \epsilon.$$

This result suggests that an analysis of the variance of the vector of differences $(s_1 - m_1, \ldots, s_N - m_N)$ might give insight into the performance of the portfolio s.

3 Proofs

Proof of Proposition 1. When $x > -1$, we can expand $\ln(1+x)$ in a Taylor's series with remainder:

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3\frac{1}{(1+\theta x)^3}, \tag{1}$$

where θ, which depends on x, satisfies $0 \le \theta \le 1$. Since

$$\gamma(x) \le \frac{1}{3}x^3\frac{1}{(1+\theta x)^3} \le \Gamma(x),$$

where the functions γ and Γ are defined by

$$\gamma(x) := \frac{1}{3}\left(\frac{x}{1+x}\right)^3, \quad \Gamma(x) := \frac{1}{3}x^3,$$

we can see that (1) implies

$$\ln(1+x) \le x - \frac{1}{2}x^2 + \Gamma(x)$$

and

$$\ln(1+x) \ge x - \frac{1}{2}x^2 + \gamma(x).$$

Notice that the functions Γ and γ are monotonically increasing.

On a few occasions we will use the identity

$$\frac{\sigma_{s-m}^2}{2} = \frac{\sigma_s^2}{2} + \frac{\sigma_m^2}{2} - \sigma_{sm}.$$

Now we are ready to start proving the proposition. First we split it into two: we will prove separately that the EMH for m predicts

$$\mu_s - \mu_m + \sigma_m^2 - \sigma_{sm} < \epsilon(1 + \sigma_{s-m}^2) \tag{2}$$

and that the EMH for m predicts

$$\mu_s - \mu_m + \sigma_m^2 - \sigma_{sm} > -\epsilon(1 + \sigma_{s-m}^2). \tag{3}$$

Indeed, if Speculator has a strategy witnessing that the EMH for m predicts (2) at level $\alpha/2$ and a strategy witnessing that the EMH for m predicts (3) at level $\alpha/2$, the combination of these strategies (i.e., splitting his money equally into two accounts and letting the first account be managed according to the first strategy and the second account be managed according to the second strategy) witnesses that the EMH for m predicts the conjunction of (2) and (3) at level α. (This argument is essentially an implicit application of the inequality

$$\underline{\mathbb{P}}(A \cap B) \geq \underline{\mathbb{P}}(A) + \underline{\mathbb{P}}(B) - 1$$

from [2], Proposition 8.10.3 on p. 186.)

First we prove (2). Without loss of generality, assume $0 < \epsilon < 1$. For any $\alpha > 0$, Speculator has a trivial strategy witnessing that the EMH for m predicts

$$\prod_{n=1}^{N} (1 + \epsilon s_n + (1 - \epsilon) m_n) < \frac{1}{\alpha} \prod_{n=1}^{N} (1 + m_n) \tag{4}$$

at level α: On each round, he invests ϵ of his capital in s (Investor's portfolio) and $1 - \epsilon$ of his capital in m. But we can rewrite (4) as

$$\sum_{n=1}^{N} \Big(\ln (1 + \epsilon s_n + (1 - \epsilon) m_n) - \ln (1 + m_n) \Big) < \ln \frac{1}{\alpha}.$$

This implies

$$\sum_{n=1}^{N} \Big(\epsilon s_n + (1 - \epsilon) m_n - \frac{1}{2} \epsilon^2 s_n^2 - \frac{1}{2} (1 - \epsilon)^2 m_n^2 - \epsilon (1 - \epsilon) s_n m_n \\ + \gamma(\epsilon s_n + (1 - \epsilon) m_n) - m_n + \frac{1}{2} m_n^2 - \Gamma(m_n) \Big) < \ln \frac{1}{\alpha}$$

or

$$\sum_{n=1}^{N} \Big(\epsilon \left(s_n - m_n + m_n^2 - s_n m_n \right) - \frac{1}{2} \epsilon^2 \left(s_n^2 + m_n^2 - 2 s_n m_n \right) \\ + \gamma(s_n \wedge m_n) - \Gamma(m_n) \Big) < \ln \frac{1}{\alpha}$$

or

$$\epsilon \left(\mu_s - \mu_m + \sigma_m^2 - \sigma_{sm} \right) - \frac{1}{2} \epsilon^2 \left(\sigma_s^2 + \sigma_m^2 - 2 \sigma_{sm} \right) \\ < \frac{1}{T} \sum_{n=1}^{N} |\gamma(s_n)| + \frac{1}{T} \sum_{n=1}^{N} |\gamma(m_n)| + \frac{1}{T} \sum_{n=1}^{N} \Gamma(m_n) + \frac{1}{T} \ln \frac{1}{\alpha}$$

or

$$\mu_s - \mu_m + \sigma_m^2 - \sigma_{sm} < \frac{1}{2}\epsilon\sigma_{s-m}^2$$
$$+ \frac{1}{3\epsilon}\sigma_s^2 \max_n \frac{|s_n|}{|1+s_n|^3} + \frac{1}{3\epsilon}\sigma_m^2 \max_n \frac{|m_n|}{|1+m_n|^3} + \frac{1}{3\epsilon}\sigma_m^2 \max_n |m_n| + \frac{1}{T\epsilon}\ln\frac{1}{\alpha}.$$

This completes the proof of (2) since $\max_n |s_n|$ and $\max_n |m_n|$ are infinitesimal and T is infinite.

Now we prove (3). Without loss of generality, assume $\epsilon < 1/3$. Consider a strategy for Speculator that calls for investing $-\epsilon$ of his capital in s and investing $1+\epsilon$ of his capital in m on every round. Using the inequalities $m_n \geq -1/2$ and $s_n \leq 1$ (remember that $\max_n |m_n|$ and $\max_n |s_n|$ are infinitesimal), we can see that this strategy's return on round n is

$$-\epsilon s_n + (1+\epsilon)m_n \geq -\epsilon + (1+\epsilon)\left(-\frac{1}{2}\right) > -1$$

and so it does not risk bankruptcy for Speculator. It witnesses that the EMH for m predicts

$$\prod_{n=1}^{N}(1 - \epsilon s_n + (1+\epsilon)m_n) < \frac{1}{\alpha}\prod_{n=1}^{N}(1+m_n) \tag{5}$$

at level α, and (5) can be transformed (analogously to (4) with ϵ replaced by $-\epsilon$) as follows:

$$\sum_{n=1}^{N}\Bigl(\ln(1 - \epsilon s_n + (1+\epsilon)m_n) - \ln(1+m_n)\Bigr) < \ln\frac{1}{\alpha};$$

$$\sum_{n=1}^{N}\Bigl(-\epsilon s_n + (1+\epsilon)m_n - \frac{1}{2}\epsilon^2 s_n^2 - \frac{1}{2}(1+\epsilon)^2 m_n^2 + \epsilon(1+\epsilon)s_n m_n$$
$$\gamma(-\epsilon s_n + (1+\epsilon)m_n) - m_n + \frac{1}{2}m_n^2 - \Gamma(m_n)\Bigr) < \ln\frac{1}{\alpha};$$

$$\sum_{n=1}^{N}\Bigl(-\epsilon\left(s_n - m_n + m_n^2 - s_n m_n\right) - \frac{1}{2}\epsilon^2\left(s_n^2 + m_n^2 - 2s_n m_n\right)$$
$$+\gamma\left(m_n \wedge (2m_n - s_n)\right) - \Gamma(m_n)\Bigr) < \ln\frac{1}{\alpha};$$

$$-\epsilon\left(\mu_s - \mu_m + \sigma_m^2 - \sigma_{sm}\right) - \frac{1}{2}\epsilon^2\sigma_{s-m}^2$$
$$< \frac{1}{T}\sum_{n=1}^{N}|\gamma(m_n)| + \frac{1}{T}\sum_{n=1}^{N}|\gamma(2m_n - s_n)| + \frac{1}{T}\sum_{n=1}^{N}\Gamma(m_n) + \frac{1}{T}\ln\frac{1}{\alpha};$$

$$\mu_s - \mu_m + \sigma_m^2 - \sigma_{sm} > -\frac{1}{2}\epsilon\sigma_{s-m}^2$$
$$-\frac{1}{3\epsilon}\sigma_m^2 \max_n \frac{|m_n|}{|1+m_n|^3} - \frac{1}{3\epsilon}\sigma_{2m-s}^2 \max_n \frac{|2m_n - s_n|}{|1+2m_n-s_n|^3} - \frac{1}{3\epsilon}\sigma_m^2 \max_n |m_n|$$

$$-\frac{1}{T\epsilon}\ln\frac{1}{\alpha}$$

(we have used the obvious notation σ^2_{2m-s}). This proves (3). $\square$

Proof of Proposition 2. We can bound the left-hand side of the inequality in Proposition 2 from above as follows:

$$\begin{aligned}
\lambda_s - \lambda_m + \frac{1}{2}\sigma^2_{s-m} \\
&\le \left(\mu_s - \frac{1}{2}\sigma^2_s + \frac{1}{T}\sum_{n=1}^{N}\Gamma(s_n)\right) - \left(\mu_m - \frac{1}{2}\sigma^2_m + \frac{1}{T}\sum_{n=1}^{N}\gamma(m_n)\right) \\
&\qquad + \frac{\sigma^2_s}{2} + \frac{\sigma^2_m}{2} - \sigma_{sm} \\
&\le \mu_s - \mu_m + \sigma^2_m - \sigma_{sm} + \frac{1}{T}\sum_{n=1}^{N}\Gamma(s_n) + \frac{1}{T}\sum_{n=1}^{N}|\gamma(m_n)|\,;
\end{aligned}$$

combining this with Proposition 1, we can see that the EMH for m predicts

$$\lambda_s - \lambda_m + \frac{1}{2}\sigma^2_{s-m} < \epsilon(1+\sigma^2_{s-m}),$$

for any $\epsilon > 0$.

In the same way, we can bound the left-hand side of the inequality in Proposition 2 from below:

$$\begin{aligned}
\lambda_s - \lambda_m + \frac{1}{2}\sigma^2_{s-m} \\
&\ge \left(\mu_s - \frac{1}{2}\sigma^2_s + \frac{1}{T}\sum_{n=1}^{N}\gamma(s_n)\right) - \left(\mu_m - \frac{1}{2}\sigma^2_m + \frac{1}{T}\sum_{n=1}^{N}\Gamma(m_n)\right) \\
&\qquad + \frac{\sigma^2_s}{2} + \frac{\sigma^2_m}{2} - \sigma_{sm} \\
&\ge \mu_s - \mu_m + \sigma^2_m - \sigma_{sm} - \frac{1}{T}\sum_{n=1}^{N}|\gamma(s_n)| - \frac{1}{T}\sum_{n=1}^{N}\Gamma(m_n);
\end{aligned}$$

combining this with Proposition 1, we can see that the EMH for m predicts

$$\lambda_s - \lambda_m + \frac{1}{2}\sigma^2_{s-m} > -\epsilon(1+\sigma^2_{s-m}),$$

for any $\epsilon > 0$. This completes the proof. $\square$

References

[1] Robert Goldblatt. *Lectures on the Hyperreals: An Introduction to Nonstandard Analysis.* Springer, New York, 1998.

[2] Glenn Shafer and Vladimir Vovk. *Probability and Finance: It's Only a Game!* Wiley, New York, 2001.

[3] Vladimir Vovk and Glenn Shafer. The Game-Theoretic Capital Asset Pricing Model. The Game-Theoretic Probability and Finance project, `http://probabilityandfinance.com`, Working Paper 1, November 2001. Journal version: *International Journal of Approximate Reasoning*, 49:175–197, 2008.

[4] Vladimir Vovk and Glenn Shafer. A probability-free and continuous-time explanation of the equity premium and CAPM. The Game-Theoretic Probability and Finance project, `http://probabilityandfinance.com`, Working Paper 44, July 2016 (first posted in June 2016).

[5] Vladimir Vovk and Glenn Shafer. Basics of a probability-free theory of continuous martingales. The Game-Theoretic Probability and Finance project, `http://probabilityandfinance.com`, Working Paper 45, March 2017 (first posted in July 2016).