Power variation and p-variation of sample functions of stochastic processes

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Power variation of a function

Let f be a regulated function on $[0,T]$, i.e. there exist limits

$$f(t-) := \lim_{x\uparrow t} f(x) \quad \text{and} \quad f(s+) := \lim_{x\downarrow s} f(x)$$

for each $0 \leq s < t \leq T$.
Let $\lambda = \{\lambda_n \colon\ n \geq 1\}$ be a nested sequence of partitions $\lambda_n = (t_i^n)_{i=0}^{m(n)}$ of $[0,T]$ such that $\cup_n \lambda_n$ is dense in $[0,T]$.
Let $1 \leq p < \infty$.
We say that f has p-th power λ-variation on $[0,T]$, if there is a regulated function V on $[0,T]$ such that $V(0) = 0$ and for each $0 \leq s < t \leq T$

$$V(t) - V(s) = \lim_{n\to\infty} \sum_{i=1}^{m(n)} |f((t_i^n \wedge t) \vee s) - f((t_{i-1}^n \wedge t) \vee s))|^p,$$

$$V(t) - V(t-) = |f(t) - f(t-)|^p \quad \text{and} \quad V(s+) - V(s) = |f(s+) - f(s)|^p.$$

p-variation of a function

Let f be a function on $[0,T]$ (must be regulated if it has bounded p-variation defined next).
Let $1 \leq p < \infty$.
The p-variation of f is the quantity $v_p(f,[0,T])$ defined to be

$$\sup\left\{\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p \colon (t_i)_{i=0}^{n} \text{ is a partition of } [0,T]\right\},$$

which may be finite or infinite.
If $v_p(f,[0,T]) < \infty$ then one says that f has bounded p-variation.
The p-variation index of f is the quantity $v(f,[0,T])$ defined to be

$$\inf\{p \geq 1 \colon \ v_p(f,[0,T]) < \infty\}.$$

if the set is non-empty and defined to be $+\infty$ otherwise.

Example: Wiener process

- Let $W = \{W(t)\colon\ t \in [0,T]\}$ be a standard *Wiener* process. Due to results of *N. Wiener* (1923) and *P. Lévy* (1940):

$$v_p(W,[0,T]) < +\infty \quad \text{a.s. iff } p > 2,$$

  and

$$v_2(W,[0,T]) = +\infty \quad \text{a.s.}$$

- Thus the p-variation index $v(W,[0,T]) = 2$ a.s.
- More precise information can be obtained in terms of ϕ-variation, defined as p-variation except that the power function $x \mapsto x^p$, $x \geq 0$, is replaced by a function ϕ.
  S. J. Taylor (1972): $v_{\psi_1}(W,[0,T]) < +\infty$ a. s., where

$$\psi_1(x) := x^2/LL(1/x), \quad 0 < x \leq e^{-e}.$$

  Also, $v_\psi(W) = +\infty$ a.s. for any ψ such that $\psi_1(x) = o(\psi(x))$ as $x \downarrow 0$.

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- Thus the p-variation index $\upsilon(W,[0,T]) = 2$ a.s.
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Example: fractional Brownian motion

Let $B_H = \{B_H(t)\colon\ t \in [0,T]\}$ be a fractional Brownian motion with the Hurst index $H \in (0,1)$, i.e. a Gaussian stochastic process with mean zero and the covariance function

$$EB_H(t)B_H(s) = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big) \quad s,t \in [0,T].$$

Let $\lambda_n = (t_i^n)_{i=0}^{m(n)}$, $n \in N$, be a sequence of partitions of $[0,T]$ such that $\big[\max_i(t_i^n - t_{i-1}^n)\big]^{1\wedge(2H)} \log n \to 0$ as $n \to \infty$. Then a.s.

$$\lim_{n\to\infty} \sum_{i=1}^{m(n)} \big|B_H(t_i^n) - B_H(t_{i-1}^n)\big|^{1/H} = E|\eta|^{1/H} T,$$

where η is a standard normal random variable.
Thus, almost every sample function of B_H has $1/H$ power λ-variation $t \mapsto c_H t$, $t \in [0,T]$.
Also, a.s. $v_{1/H}(B_H, [0,T]) = +\infty$ and $\upsilon(B_H, [0,T]) = 1/H$.

Weighted power variation for a Gaussian process

- Let $X = \{X(t)\colon\ t \in [0,T]\}$ be a mean zero Gaussian process s.t. there is a real valued function ρ defined on $[0,T]$ and "equivalent" to
$$h \mapsto (E[X(s+h) - X(s)]^2)^{1/2}$$
near zero uniformly in $s \in [\epsilon, T)$ for each $\epsilon > 0$.
If X has stationary increments, then one can take
$$\rho(h) = (E[X(s+h) - X(s)]^2)^{1/2}.$$

- Under suitable hypotheses on the covariance of X and for a suitable set of positive r we proved that a.s.
$$\lim_{n\to\infty} \sum_{i=1}^{m_n} \frac{|X(t_i^n) - X(t_{i-1}^n)|^r}{[\rho(t_i^n - t_{i-1}^n)]^r}(t_i^n - t_{i-1}^n) = E|\eta|^r T, \qquad (1)$$
where η is a standard normal random variable, and $((t_i^n)_{i=0}^{m_n})$ is a sequence of partitions of $[0,T]$ such that the mesh $\max_i(t_i^n - t_{i-1}^n)$ tends to zero as $n \to \infty$ sufficiently fast.

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Partial sum process

Let $X_1, X_2, \dots$ be real random variables. For each $n = 1, 2, \dots,$ let S_n be the n-th partial sum process

$$S_n(t) := X_1 + \cdots + X_{\lfloor tn \rfloor}, \quad t \in [0,1],$$

Thus for each $n = 1, 2, \dots$ and $t \in [0,1]$,

$$S_n(t) = \begin{cases} 0, & \text{if } t \in [0, 1/n), \\ X_1 + \cdots + X_k, & \text{if } t \in [\frac{k}{n}, \frac{k+1}{n}), \\ & k \in \{1, \dots, n-1\}, \\ X_1 + \cdots + X_n, & \text{if } t = 1. \end{cases}$$

Then for any $p \in (0, \infty)$,

$$v_p(S_n, [0,1]) = \max \left\{ \sum_{j=1}^{m} \left| X_{k_{j-1}+1} + \cdots + X_{k_j} \right|^p \right\},$$

where the maximum is taken over $0 = k_0 < \cdots < k_m = n$ and $1 \le m \le n$.

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p-variation of partial sum process

- *J. Bretagnolle* (1972): given $p \in (0, 2)$ there exists a finite constant C_p such that

$$\left(\sum_{i=1}^{n} E|X_i|^p \leq \right) Ev_p(S_n) \leq C_p \sum_{i=1}^{n} E|X_i|^p,$$

  provided $X_1, X_2, \dots$ are independent, $E|X_i|^p < \infty$ and $EX_i = 0$ if $p > 1$.

- Suppose that $X_1, X_2, \dots$ are independent identically distributed real random variables, $EX_1 = 0$ and $EX_1^2 = 1$. Let $Lx := \max\{1, \log x\}$, $x > 0$.

  J. Qian (1998): boundedness in probability

$$v_2(S_n) = O_P(nLLn) \quad \text{as } n \to \infty.$$

- Also, $O_P(nLLn)$ cannot be replaced by $o_p(nLLn)$, if in addition $E|X_1|^{2+\epsilon} < \infty$ for some $\epsilon > 0$.

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- *R. Norvaiša* and *A. Račkauskas* (2008): Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables and let S_n be the n-th partial sum process.
  The convergence in law (in the sense of *Hoffmann-Jørgensen*)
  $$n^{-1/2}S_n \Rightarrow \sigma W \quad \text{in } \mathcal{W}_p[0,1] \text{ as } n \to \infty$$
  holds if and only if
  $$EX_1 = 0 \text{ and } \sigma^2 := EX_1^2 < \infty.$$
  Here $\mathcal{W}_p[0,1]$ is the Banach space of functions f on $[0,1]$ having bounded p-variation with respect to the norm
  $$\|f\|_{[p]} := \|f\|_{\sup,[0,1]} + v_p(f,[0,1])^{1/p}.$$
- In particular, the convergence in distribution
  $$n^{-p/2}v_p(S_n,[0,1]) \to \sigma^p v_p(W,[0,1]) \quad \text{as } n \to \infty$$
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For a comparison

- It is interesting to compare this fact with the related convergence of smoothed partial sum processes with respect to the α-Hölder norm. Let $\tilde{S}_n$ be a (random) function obtained from S_n by linear interpolation between points

$$\left(\frac{k}{n}, S_n\left(\frac{k}{n}\right)\right) \quad \text{ir} \quad \left(\frac{k+1}{n}, S_n\left(\frac{k+1}{n}\right)\right)$$

  $k = 0, 1, \ldots, n-1$.

- *A. Račkauskas* and *C. Suquet* (2004): Let $p > 2$. Convergence in law

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Why we do what we do?

To develop calculus without probability:

- analysis of integral equations with respect to rough functions (having unbounded variation);
- analysis of nonlinear functionals and operators acting on the Banach space of functions of bounded p-variation;
- statistical analysis of the index of p-variation for sample functions of various stochastic processes.

Some publications:

[1] R.M. Dudley and R.N. Differentiability of Six Operators on Nonsmooth Functions and p-variation. Lecture Notes in Mathematics, vol. 1703, 1999.

[2] R.N. Quadratic variation, p-variation and integration with applications to stock price modelling. arXiv: 010890[math.CA], 2001.

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